Ludwig-Maximilians-Universität München
Department of Mathematics

Master Thesis
Mathematics

Brownian Motion and the Dirichlet Problem

Brownsche Bewegung und das Dirichlet-Problem

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"The scientist finds his reward in what Henri Poincaré calls the joy of comprehension, and not in the possibilities of application to which any discovery of his may lead."

"Der Wissenschaftler findet seine Belohnung in dem, was Poincaré die Freude am Verstehen nennt, nicht in den Anwendungsmöglichkeiten seiner Erfindung."

- Albert Einstein

Abstract

Since its mathematical formulation, Brownian motion is the most important stochastic process in probability theory. In the Fourties and Fifties, pioneers as Kakutani and Doob did some research in the area of stochastical analysis and recognized the importance of Brownian motion for the Dirichlet problem.

This master thesis intends to give an explanation of the connection of Brownian motion and the Dirichlet problem in a modern, mathematical and closed context.

Although the results can be further generalized to Markov processes, in case of Brownian motion and its useful properties, they stay clear and understandable which effects the corresponding proofs.

Zusammenfassung

Seit ihrer mathematischen Formulierung ist die Brownsche Bewegung der wichtigste stochastische Prozess der Wahrscheinlichkeitstheorie. In den 1940er und 1950er Jahren forschten Pioniere wie Kakutani und Doob im Gebiet der stochastischen Analysis und erkannten die Bedeutung der Brownschen Bewegung für das Dirichlet-Problem.

Diese Masterarbeit beabsichtigt die Behandlung und Erklärung des Zusammenhangs der Brownschen Bewegung und des Dirichlet-Problems in einem modernen, mathematischen und abgeschlossenen Kontext.

Obwohl die Ergebnisse noch weiter auf Markov-Prozesse abstrahiert und verallgemeinert werden können, bleiben sie im Falle der Brownschen Bewegung und ihrer nützlichen Eigenschaften übersichtlich und verständlich, was sich in den zugehörigen Beweisen bemerkbar macht.

Preface

This thesis represents the end of my master studies in Mathematics at the LMU Munich. The topic was offered by my supervisor Prof. Dr. Gregor Svindland (LMU Munich, Department of Mathematics) and came out of a little conversation about possible topics which could fit for a master thesis.

The title of this thesis is *Brownian Motion and the Dirichlet Problem* which is a subject in a small but beautiful area of stochastic analysis, the area in which stochastics and analysis touch. My thesis is divided into two chapters. The first one is a preparation for the main result(s) in the second chapter. It provides a modern, mathematical setting and contains important basic definitions and first results which will be used to prove the two main outcomes, the *Markov property* and the *strong Markov property* of one-dimensional Brownian motion. The second chapter is the main part of this thesis in which we use the properties of Brownian motion to establish a connection to the Dirichlet problem. In the beginning, we prove some basic PDE results to provide a suitable framework good to work with. Afterwards, we prove the first main result concerning the Dirichlet problem on bounded domains which has a nice application for the path behavior of (multi-dimensional) Brownian motion. In the end, we treat the Dirichlet problem on unbounded domains and additionally find a probabilistic result to a boundary value problem for the Poisson equation.

Throughout the whole thesis, I worked a lot with Liggett's book *Continuous Time Markov Processes* ([**1**]). This text gave me the most inspiration, so I used many results, ideas and examples from it. The PDE results in the first section of the second chapter were heavily influenced by Evans' *Partial Differential Equations* ([**2**]). In some places, I used [**3**] and [**4**], but these are rather exceptional.

All this was fun! I would like to thank Prof. Dr. Gregor Svindland for his support and answering my questions in several sessions. Furthermore, my whole master studies would not have been possible without the *German National Academic Foundation* which supported me with a scholarship the whole time. I really appreciate their trust and their help which makes me feel responsible for the future to give something back.

This thesis is dedicated to all people who believe in me. First of all, I would like to thank my family for their support during my whole life. But I would not be the same character as I am today without my friends. All the times we spent talking, joking, laughing and crying made me the person I am today.

Thank you!

Mario Teixeira Parente Munich, April 2, 2016

Contents

Chapter 1

Preparation: One-Dimensional Brownian Motion

The first chapter intends to give an introduction, in which we will define a suitable context, and wants to prepare some important results for the main chapter — Chapter 2. There will be some results which seem special and not that important at first sight, but they will play a significant role at some point later. The two main outcomes of this chapter are *the Markov property* and *the strong Markov property* for one-dimensional Brownian motion. In general, they have a huge amount of applications; we will need them to deal with harmonicity in the second chapter.

1.1 Basics

1.1.1 Setting

As mentioned, we need a suitable context, in which we are doing our mathematics. The most important object — Brownian motion — has to be defined clearly, and afterwards we have to choose a suitable probability space on which the Brownian motion can live.

Definition 1.1. A stochastic process $(B_t)_{t\geq 0}$ is called a *standard Brownian motion* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if it satisfies the following properties:

(i) $B_0 = 0 \quad \mathbb{P}$-a.s.

(ii) $t \mapsto B_t(\omega)$ is continuous for every $\omega \in \Omega$.

(iii) If $0 \leq s < t$, then $B_t - B_s \sim \mathcal{N}(0, t-s)$.

(iv) For $0 \leq t_0 < t_1 < \ldots < t_k$, the increments $\left(B_{t_{i+1}} - B_{t_i}\right)_{i=0,\ldots,k-1}$ are independent.

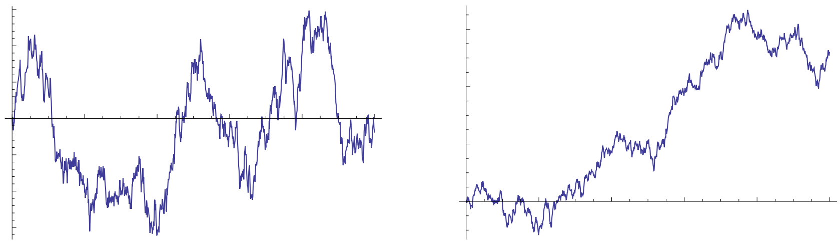

Figure 1.1: Two sample paths of one-dimensional Brownian motion

Since Brownian motion is defined only by distributional properties, we are free in our choice of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for it. There are many possibilities for this choice.

We decide to use the so-called *canonical model*. Here, the sample space Ω is chosen to be

$$C_0[0,\infty) := \{\omega : [0,\infty) \to \mathbb{R} : \omega \text{ is continuous and } \omega(0) = 0\},$$

which is the set of all $\mathbb{R}$-valued continuous functions on $[0,\infty)$ starting at 0. This choice is natural and makes perfect sense, since the paths of Brownian motion (sometimes also called "Brownian paths") are defined to be continuous and start in 0. For the corresponding σ-algebra $\mathcal{F}$, let us first define the projection function, which is

$$\pi_t : \Omega \to \mathbb{R},\ \omega \mapsto \omega(t), \qquad t \geq 0. \tag{1.1}$$

This function simply evaluates a Brownian path at time $t \geq 0$. $\mathcal{F}$ is defined as the smallest σ-algebra for which the projection π_t is measurable for each $t \geq 0$. The probability measure $\mathbb{P}$ is chosen in such a way that the stochastic process $(B_t)_{t\geq 0}$, defined as

$$B : [0,\infty) \times \Omega \to \mathbb{R},\ (t,\omega) \mapsto \omega(t)$$

becomes a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. There is such a probability measure, called the *Wiener measure*, which can be constructed in a formal way — see [3].

In almost every case dealing with Brownian motion, we have to consider an arbitrary starting point $x \in \mathbb{R}$, which can be different from 0. For this, we introduce a whole new family of probability measures $(\mathbb{P}^x)_{x\in\mathbb{R}}$ on $(\Omega, \mathcal{F})$. A probability measure

$\mathbb{P}^x$, $x \in \mathbb{R}$, is defined to be the distribution of the random variable $t \mapsto B_t(\cdot) + x$, i.e.

$$\mathbb{P}^x(A) := \mathbb{P}\left((t \mapsto B_t + x) \in A\right), \quad A \in \mathcal{F}, \tag{1.2}$$

where B is a standard Brownian motion. Hence, for $x \in \mathbb{R}$, it holds that

$$B_0 = x \quad \mathbb{P}^x\text{-a.s.},$$

which means that Brownian motion is starting from x under the probability measure $\mathbb{P}^x$. It remains to mention that the expectation under $\mathbb{P}^x$ is denoted by $\mathbb{E}^x$.

Before dealing with the (strong) Markov property, we need the notion of a filtration $(\mathcal{F}_t)_{t\geq 0}$ which, from a mathematical point of view, is an increasing family of sub-σ-algebras of $\mathcal{F}$. $\mathcal{F}_t$ can be interpreted as the information we have about the process at time $t \geq 0$. In our case, it is natural to define $\mathcal{F}_t^0$, $t \geq 0$, as the smallest σ-algebra with respect to which the projection π_s is measurable for each $0 \leq s \leq t$. Unfortunately, the filtration $(\mathcal{F}_t^0)_{t\geq 0}$ is not right-continuous, but this property becomes very important later. We can solve this problem by defining

$$\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}_s^0, \qquad t \geq 0.$$

Indeed, we get

$$\bigcap_{s>t} \mathcal{F}_s = \bigcap_{s>t} \Big(\bigcap_{r>s} \mathcal{F}_r^0 \Big) = \bigcap_{r>t} \mathcal{F}_r^0 = \mathcal{F}_t,$$

which shows the right-continuity of $\mathcal{F}_t$ and completes setting up our mathematical context.

Beside the Markov property, there is another notion that has become very important for probability theory: The notion of a *martingale*.

Definition 1.2. Let $\mathbb{P}$ be a probability measure and $(\mathcal{G}_t)_{t\geq 0}$ a filtration on some probability space. A family of integrable adapted random variables $(M_t)_{t\geq 0}$ is called a *martingale*, if it holds for $0 \leq s < t$

$$\mathbb{E}\left[M_t \,|\, \mathcal{G}_s\right] = M_s \quad \mathbb{P}\text{-a.s.}$$

The next proposition lists some martingales constructed with Brownian motion. We will need them later in Chapter 2.

Proposition 1.3. *Let $x \in \mathbb{R}$ and assume that all B's occurring below are standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. Then it holds (each time under $\mathbb{P}^x$):*

(a) $(B_t^2 - t)_{t\geq 0}$ *is a martingale with respect to* $(\mathcal{F}_t)_{t\geq 0}$.

(b) If $(\tilde{B}_t)_{t\geq 0}$ *and* $(\hat{B}_t)_{t\geq 0}$ *are independent, then* $(\tilde{B}_t\hat{B}_t)_{t\geq 0}$ *is a martingale with respect to the (right-continuous) filtration generated by the two processes.*

(c) $(B_t^4 - 6tB_t^2 + 3t^2)_{t\geq 0}$ *is a martingale with respect to* $(\mathcal{F}_t)_{t\geq 0}$.

Remark. For the proof, recall that for a Brownian motion $(B_t)_{t\geq 0}$ and $0 \leq s < t$, $B_t - B_s$ is independent of $\mathcal{F}_s$.

Proof of Proposition 1.3. (a): For $t \geq 0$,

$$\mathbb{E}^x \left| B_t^2 - t \right| \leq \underbrace{\mathbb{E}^x B_t^2}_{=t} + t = 2t < \infty$$

shows the integrability. Obviously, it is adapted to $(\mathcal{F}_t)_{t\geq 0}$. For the martingale property, let $0 \leq s < t$ and compute

$$\begin{aligned}
\mathbb{E}^x \left[B_t^2 - t \,\middle|\, \mathcal{F}_s \right] &= \mathbb{E}^x \left[(B_t - B_s + B_s)^2 \,\middle|\, \mathcal{F}_s \right] - t \\
&= \mathbb{E}^x \left[(B_t - B_s)^2 \,\middle|\, \mathcal{F}_s \right] + 2\,\mathbb{E}^x \left[(B_t - B_s)B_s \,|\, \mathcal{F}_s \right] + \mathbb{E}^x \left[B_s^2 \,\middle|\, \mathcal{F}_s \right] - t \\
&\overset{(*)}{=} \underbrace{\mathbb{E}^x \left[(B_t - B_s)^2 \right]}_{=t-s} + 2B_s \underbrace{\mathbb{E}^x \left[B_t - B_s \right]}_{=0} + B_s^2 - t \\
&= t - s + B_s^2 - t = B_s^2 - s \quad \mathbb{P}^x\text{-a.s.},
\end{aligned}$$

where we used the independence of the increments at $(*)$.

(b): Integrability: For $t \geq 0$,

$$\mathbb{E}^x \tilde{B}_t \hat{B}_t = \mathbb{E}^x \tilde{B}_t \,\mathbb{E}^x \hat{B}_t = x^2 < \infty.$$

Let $(\mathcal{G}_t)_{t\geq 0}$ be the right-continuous filtration generated by $(\tilde{B}_t)_{t\geq 0}$ and $(\hat{B}_t)_{t\geq 0}$. Adaptedness is obvious. Martingale property: For $0 \leq s < t$, we get

$$\begin{aligned}
\mathbb{E}^x \left[\tilde{B}_t \hat{B}_t \,\middle|\, \mathcal{G}_s \right] &= \mathbb{E}^x \left[(\tilde{B}_t - \tilde{B}_s + \tilde{B}_s)\hat{B}_t \,\middle|\, \mathcal{G}_s \right] \\
&= \mathbb{E}^x \left[(\tilde{B}_t - \tilde{B}_s)\hat{B}_t \,\middle|\, \mathcal{G}_s \right] + \underbrace{\mathbb{E}^x \left[\tilde{B}_s \hat{B}_t \,\middle|\, \mathcal{G}_s \right]}_{=\tilde{B}_s\hat{B}_s} \\
&= \mathbb{E}^x \left[(\tilde{B}_t - \tilde{B}_s)(\hat{B}_t - \hat{B}_s + \hat{B}_s) \,\middle|\, \mathcal{G}_s \right] + \tilde{B}_s\hat{B}_s \\
&= \mathbb{E}^x \left[(\tilde{B}_t - \tilde{B}_s)(\hat{B}_t - \hat{B}_s) \,\middle|\, \mathcal{G}_s \right] + \mathbb{E}^x \left[(\tilde{B}_t - \tilde{B}_s)\hat{B}_s \,\middle|\, \mathcal{G}_s \right] + \tilde{B}_s\hat{B}_s \\
&= \underbrace{\mathbb{E}^x \left[\tilde{B}_t - \tilde{B}_s \right]}_{=0} \underbrace{\mathbb{E}^x \left[\hat{B}_t - \hat{B}_s \right]}_{=0} + \hat{B}_s \underbrace{\mathbb{E}^x \left[\tilde{B}_t - \tilde{B}_s \right]}_{=0} + \tilde{B}_s\hat{B}_s \\
&= \tilde{B}_s\hat{B}_s \quad \mathbb{P}^x\text{-a.s.},
\end{aligned}$$

where we used the independence of the increments again.

(c): The process is integrable, since

$$\mathbb{E}^x\left|B_t^4 - 6tB_t^2 + 3t^2\right| \leq \underbrace{\mathbb{E}^x B_t^4}_{=3t^2} + 6t \underbrace{\mathbb{E}^x B_t^2}_{=t} + 3t^2 = 12t^2 < \infty$$

for $t \geq 0$. Adaptedness is again obvious. Using the independence of the increments, we can show the martingale property by

$$\begin{aligned}
&\mathbb{E}^x\left[B_t^4 - 6tB_t^2 + 3t^2 \,\middle|\, \mathcal{F}_s\right] \\
&\quad = \mathbb{E}^x\left[(B_t - B_s + B_s)^4 \,\middle|\, \mathcal{F}_s\right] - 6t\,\mathbb{E}^x\left[(B_t - B_s + B_s)^2 \,\middle|\, \mathcal{F}_s\right] + 3t^2 \\
&\quad = \mathbb{E}^x\left[(B_t - B_s)^4 + 4(B_t - B_s)^3 + 6(B_t - B_s)^2 B_s^2 + 4(B_t - B_s)B_s^3 + B_s^4 \,\middle|\, \mathcal{F}_s\right] \\
&\qquad - 6t\,\mathbb{E}^x\left[(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 \,\middle|\, \mathcal{F}_s\right] + 3t^2 \\
&\quad = \underbrace{\mathbb{E}^x\left[(B_t - B_s)^4\right]}_{=3(t-s)^2} + 4B_s \underbrace{\mathbb{E}^x\left[(B_t - B_s)^3\right]}_{=0} + 6B_s^2 \underbrace{\mathbb{E}^x\left[(B_t - B_s)^2\right]}_{=t-s} + 4B_s^3 \underbrace{\mathbb{E}^x\left[B_t - B_s\right]}_{=0} + B_s^4 \\
&\qquad - 6t\Big(\underbrace{\mathbb{E}^x\left[(B_t - B_s)^2\right]}_{=t-s} + 2B_s \underbrace{\mathbb{E}^x\left[B_t - B_s\right]}_{=0} + B_s^2\Big) + 3t^2 \\
&\quad = 3t^2 - 6st + 3s^2 + 6(t-s)B_s^2 + B_s^4 - 6t^2 + 6st - 6tB_s^2 + 3t^2 \\
&\quad = B_s^4 - 6sB_s^2 + 3s^2 \quad \mathbb{P}^x\text{-a.s.}
\end{aligned}$$

for each $s, t \in \mathbb{R}$ with $0 \leq s < t$. ■

1.1.2 Hitting times

This subsection introduces the notion of a *stopping time*, which will help us in many situations and is a necessary ingredient for stating the strong Markov property.

Definition 1.4. A random variable $\tau : \Omega \to [0, \infty)$ is called a *stopping time* (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$), if it holds

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

A *hitting time* is a special type of a stopping time. It will play an important role, when dealing with bounded domains related to the Dirichlet problem in Chapter 2. Proposition 1.5 and 1.7 deal with hitting times of open and closed sets.

Proposition 1.5. *Let $G \subset \mathbb{R}$ be open and define $\tau := \inf\{t > 0 : B_t \in G\}$. Then τ is a stopping time.*

Proof. We need to prove that

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

Note that it suffices to show that

$$\{\tau < t\} \in \mathcal{F}_t \quad \text{for all } t \geq 0.$$

Indeed, for $t \geq 0$, we see that

$$\{\tau \leq t\} = \bigcap_{n\in\mathbb{N}} \left\{\tau < t + \frac{1}{n}\right\} \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t.$$

So, let $t > 0$. We claim that

$$\{\tau < t\} = \bigcup_{\substack{0\leq s<t \\ s\in\mathbb{Q}}} \{B_s \in G\}.$$

Indeed, if $\tau(\omega) < t$, there is some $0 \leq r < t$, $r \in \mathbb{R}$, such that $B_r \in G$. Since G is open and the sample paths are continuous, we find some $r < s < t$, $s \in \mathbb{Q}$, with $B_s \in G$. Thus, $\omega \in \bigcup_{\substack{0\leq s<t \\ s\in\mathbb{Q}}} \{B_s \in G\}$.

Conversely, let $\omega \in \{B_s \in G\}$ for some $0 \leq s < t$, $s \in \mathbb{Q}$. Since G is open and the sample paths are continuous, we get $\tau(\omega) \leq s < t$.

For completeness, if $t = 0$, then

$$\{\tau < 0\} = \emptyset \in \mathcal{F}_0.$$

■

Lemma 1.6. *If $(\tau_n)_{n\in\mathbb{N}}$ is an increasing or decreasing sequence of stopping times such that $\tau_n \xrightarrow{n\to\infty} \tau$, then τ is again a stopping time.*

Proof. First, let $(\tau_n)_{n\in\mathbb{N}}$ be increasing, i.e. $\tau_n \leq \tau$ for every $n \in \mathbb{N}$. Then, it holds

$$\{\tau \leq t\} = \bigcap_{n\in\mathbb{N}} \underbrace{\{\tau_n \leq t\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t$$

for all $t \geq 0$.

Now, let $(\tau_n)_{n\in\mathbb{N}}$ be decreasing, i.e. $\tau_n \geq \tau$ for every $n \in \mathbb{N}$. Then, it holds

$$\{\tau > t\} = \bigcap_{n\in\mathbb{N}} \underbrace{\{\tau_n > t\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t$$

for all $t \geq 0$. It follows that $\{\tau \leq t\} = \{\tau > t\}^c \in \mathcal{F}_t$ for all $t \geq 0$. ■

Proposition 1.7. *Let $F \subset \mathbb{R}$ be closed and define $\tau := \inf\{t > 0 : B_t \in F\}$. Then τ is a stopping time.*

Proof. We approximate F by a sequence of open neighborhoods from the outside. For $n \in \mathbb{N}$, define

$$G_n := \left\{ x \in \mathbb{R} : |x - y| < \frac{1}{n} \text{ for some } y \in F \right\}$$

and let τ_n be the hitting time of G_n. Note that τ_n is a stopping time for every $n \in \mathbb{N}$ by Proposition 1.5 and that the sequence $(\tau_n)_{n\in\mathbb{N}}$ is increasing, since G_n is decreasing. Hence, by Lemma 1.6, $\sigma := \lim_{n\to\infty} \tau_n$ is again a stopping time. Furthermore, note that $\sigma \leq \tau$, since $\tau_n \leq \tau$ for each $n \in \mathbb{N}$. So, it suffices to show that $B_\sigma \in F$ implying $\tau \leq \sigma$, and thus $\tau = \sigma$. In doing so, we only regard the event $\{\sigma < \infty\}$, since $\tau \leq \sigma$ is clear on the complementary event. If $k \leq m$, then

$$B_{\tau_m} \in \overline{G_m} \subset \overline{G_k}.$$

It follows by path continuity,

$$B_\sigma = \lim_{n\to\infty} B_{\tau_n} \in \bigcap_{n\in\mathbb{N}} \overline{G_n} = F,$$

where the last equality comes from the closedness of F. Hence, $\tau = \sigma$ implying that τ is a stopping time. ■

Remark. (a) A stopping time defined as τ above in Proposition 1.5 and 1.7 is called a *hitting time of* G, or F respectively.
(b) This chapter deals with Brownian motion in one dimension, but the proofs of Proposition 1.5 and 1.7 also apply well to Brownian motion with values in a metric space.

1.2 The Markov property

Before taking care of the Markov property itself, we have to introduce some rather technical instruments and results. The next definition brings random variables of a specific form and particular sets of $\mathcal{F}$ into play. In this section and also in the next one, we will prove some statements only for these kinds of random variables and sets. The monotone class theorem — Theorem A.8 — will allow us to generalize the statements suitably.

Definition 1.8. A random variable X is called *special*, if it is of the form

$$X(\omega) = \prod_{m=1}^{n} f_m(\omega(t_m)), \tag{1.3}$$

where $0 < t_1 < \ldots < t_n$ and $f_1, \ldots, f_n$ are continuous functions on $\mathbb{R}$ tending to 0 at $\pm\infty$. Correspondingly, $A \in \mathcal{F}$ is said to be *finite-dimensional*, if there exist points in time $0 < r_1 < \ldots < r_k$ and open subsets $A_1, \ldots, A_k$ of $\mathbb{R}$ such that

$$A = \{\omega \in \Omega : \omega(r_1) \in A_1, \ldots, \omega(r_k) \in A_k\}. \tag{1.4}$$

Note that $\mathcal{F}$ is the smallest σ-algebra containing all finite-dimensional sets.

The next few lines are dedicated to show that the generalization via the monotone class theorem indeed works. First, note that the appearing π-systems $\mathcal{P}$, containing finite-dimensional sets, indeed fulfill the properties of a π-system (Definition A.6), since the intersection of two finite-dimensional sets is again finite-dimensional by ordering the corresponding points in time. Additionally, $\Omega \in \mathcal{P}$ is required by the monotone class theorem, but this case is obvious (choose $A_1 = \mathbb{R}$).

The sets $\mathcal{H}$ in the upcoming proofs will always contain *bounded* random variables fulfilling the statement we want to show. The vector space properties and property (ii) of the monotone class theorem can be easily shown for all appearing $\mathcal{H}$'s on the next pages. The interesting part for $\mathcal{H}$ to prove is property (i):

$$\mathbb{1}_A \in \mathcal{H} \quad \text{for finite-dimensional } A \in \mathcal{F}. \tag{1.5}$$

So, take a finite-dimensional $A \in \mathcal{F}$. Observe that

$$\mathbb{1}_A(\omega) = \prod_{m=1}^{n} \mathbb{1}_{A_m}(\omega(t_m)).$$

Furthermore, note that each $\mathbb{1}_{A_m}$, $m \in [n]$, can be approximated by an increasing sequence of continuous function $(f_m^k)_{k\in\mathbb{N}}$ tending to 0 at $\pm\infty$, for example

$$f_m^k(x) := 1 \wedge k \operatorname{dist}(x, A_m^c) \xrightarrow{k\to\infty} \mathbb{1}_{A_m}(x),$$

where $\operatorname{dist}(x, S) := \inf\{|x-y| : y \in S\}$ is continuous in x for every open set S. Note that if $A_m = \mathbb{R}$ for some $m \in [n]$, then $\operatorname{dist}(x, A_m^c) = \inf \emptyset = \infty$. This yields to

$$\mathbb{1}_A(\omega) = \lim_{k\to\infty} \prod_{m=1}^{n} f_m^k(\omega(t_m)). \tag{1.6}$$

Hence, if we can prove $X \in \mathcal{H}$ for a special random variable X, we know that (1.5) holds by (1.6) and property (ii) of the monotone class theorem.

So, whenever we use the monotone class theorem on the next pages, it suffices to show the relevant statement for special random variables and finite-dimensional sets, and to prove that the corresponding $\mathcal{H}$ satisfies the properties required.

Proposition 1.9. *Let X be a bounded random variable. Then the function*

$$\mathbb{R} \ni x \mapsto \mathbb{E}^x X$$

is measurable.

Proof. As mentioned above, the plan is to prove the statement for special X, and then to extend it to bounded X via the monotone class theorem.

Take $x \in \mathbb{R}$ and a special random variable X. Write

$$\mathbb{E}^x X = \mathbb{E}^x \prod_{m=1}^{n} f_m(\omega(t_m)) = \mathbb{E} \prod_{m=1}^{n} f_m(x + B_{t_m}).$$

We show that this expression is continuous in x by induction on n, which implies measurability. For $t > 0$, write $p_t(x, \cdot)$ for the density of the $\mathcal{N}(x,t)$-distribution. For $n = 1$, we get

$$\mathbb{E}^x X = \mathbb{E} f_1(x + B_{t_1}) = \int_{\mathbb{R}} f_1(x+y) p_{t_1}(0,y)\, dy = \int_{\mathbb{R}} f_1(z) p_{t_1}(x,z)\, dz, \tag{1.7}$$

which is continuous in x. For the induction step, recall the statement of Proposition A.11 and use the independence of Brownian increments to get

$$\begin{aligned}
\mathbb{E}^x X &= \mathbb{E} \prod_{m=1}^{n+1} f_m(x + B_{t_m}) \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{m=1}^{n} f_m(x + B_{t_m}) \cdot f_{n+1}(x + B_{t_{n+1}}) \,\middle|\, B_{t_i}, 1 \le i \le n \right]\right] \\
&= \mathbb{E}\Bigg[\prod_{m=1}^{n} f_m(x + B_{t_m}) \\
&\qquad \cdot \mathbb{E} f_{n+1}(x + B_{t_n} + B_{t_{n+1}} - B_{t_n}) \,\Bigg|\, B_{t_i}, 1 \le i \le n \Bigg] \\
&= \mathbb{E}\left[\prod_{m=1}^{n} f_m(x + B_{t_m}) \cdot g(x + B_{t_n}) \right],
\end{aligned} \tag{1.8}$$

where

$$g(u) := \mathbb{E} f_{n+1}(u + B_{t_{n+1} - t_n}),$$

which is continuous by (1.7). By the induction hypothesis, the right side of (1.8) is continuous in x, which completes the induction step.

Finally, to extend the result to bounded random variables, we apply the monotone class theorem. Therefore, let the π-system $\mathcal{P}$ contain all finite-dimensional sets and set

$$\mathcal{H} := \{X \text{ bounded} : \mathbb{R} \ni x \mapsto \mathbb{E}^x X \text{ is measurable}\}.$$

The vector space properties are obvious. Property (ii) of the monotone class theorem is also obvious by using the bounded convergence theorem. Property (i) was shown as a special case above. So, $\mathcal{H}$ contains all bounded random variables fulfilling the desired statement, which completes the proof. ■

The final missing piece to state our version of the Markov property is the *time-shift-operator*. It shifts the time for a Brownian path from t to $t+s$ for $s, t \geq 0$, and is defined as

$$\theta_s : \Omega \to \Omega,\ \omega \mapsto (t \mapsto \omega(t+s)),$$

implying $\theta_s(\omega)(t) = \omega(t+s) = B_{t+s}(\omega)$. Recall that $(B_t \circ \theta_s - B_s)_{t \geq 0}$ is again a (standard) Brownian motion for $s \geq 0$, which is another version of the Markov property. Our version states, informally said, that if we want to compute the conditional expectation of a bounded random variable X, time-shifted by $s \geq 0$, while knowing the Brownian path ω up to time s, we can also start a new Brownian motion at $B_s(\omega)$ and compute the ordinary expectation value using the probability measure $\mathbb{P}^{B_s(\omega)}$. The formal statement of this is given by the next theorem.

Theorem 1.10 (Markov Property). *Let X be a bounded random variable. Then for every $x \in \mathbb{R}$ and $s \geq 0$, it holds*

$$\mathbb{E}^x [X \circ \theta_s \,|\, \mathcal{F}_s] = \mathbb{E}^{B_s} X \quad \mathbb{P}^x\text{-a.s.} \tag{1.9}$$

Remark. (a) The right side of (1.9) means the composition of the function $y \mapsto \mathbb{E}^y X$ with B_s, i.e.

$$\mathbb{E}^{B_s}[X](\omega) := \int_\Omega X(\omega')\, \mathbb{P}^{B_s(\omega)}(d\omega').$$

(b) The right side of (1.9) is $\mathcal{F}_s^0$-measurable by Proposition 1.9. Since the left side is $\mathcal{F}_s$-measurable and $\mathcal{F}_s^0 \subset \mathcal{F}_s$, (1.9) is a stronger statement than

$$\mathbb{E}^x \left[X \circ \theta_s \,\middle|\, \mathcal{F}_s^0\right] = \mathbb{E}^{B_s} X \quad \mathbb{P}^x\text{-a.s.} \tag{1.10}$$

Proof of Theorem 1.10. Let $x \in \mathbb{R}$ and $s \geq 0$. We have to show that the right side of (1.9) satisfies the defining property of the conditional expectation on the left, i.e. we need to prove

$$\mathbb{E}^x [X \circ \theta_s, A] = \mathbb{E}^x \left[\mathbb{E}^{B_s} X, A\right] \tag{1.11}$$

for all bounded random variables X and all $A \in \mathcal{F}_s$. Again, we first consider special X and finite-dimensional sets $A \in \mathcal{F}_s$ as in (1.3) and (1.4), where

$$0 < r_1 < \ldots < r_k < s + h < s + t_1 < \ldots < s + t_n.$$

We choose $0 < h < t_1$, since we want to prove (1.9) in the end instead of (1.10). To show (1.11), we first want to prove

$$\mathbb{E}^x [X \circ \theta_s, A] = \mathbb{E}^x \left[\mathbb{E}^{B_{s+h}} \left[\prod_{m=1}^{n} f_m(\omega(t_m - h))\right], A\right]. \tag{1.12}$$

For simplicity, let $k = n = 1$ and calculate

$$\begin{aligned}
\mathbb{E}^x [X \circ \theta_s, A] &= \mathbb{E}^x [f_1(\omega(t_1 + s)), \omega(r_1) \in A_1] \\
&= \int_{A_1} p_{r_1}(x, u) \cdot \mathbb{E}^u f_1(\omega(t_1 + s - r_1))\, du \\
&= \int_{A_1} p_{r_1}(x, u) \left(\int_{\mathbb{R}} p_{t_1+s-r_1}(u, z) f_1(z)\, dz\right) du \\
&\overset{(*)}{=} \int_{A_1} p_{r_1}(x, u) \int_{\mathbb{R}} \left(\int_{\mathbb{R}} p_{s+h-r_1}(u, v)\, p_{t_1-h}(v, z)\, dv\right) f_1(z)\, dz\, du \\
&\overset{(**)}{=} \int_{A_1} p_{r_1}(x, u) \Bigg(\int_{\mathbb{R}} p_{s+h-r_1}(u, v) \underbrace{\left(\int_{\mathbb{R}} p_{t_1-h}(v, z) f_1(z)\, dz\right)}_{=\mathbb{E}^v f_1(\omega(t_1-h))} dv\Bigg) du \\
&= \mathbb{E}^x \left[\mathbb{E}^{B_{s+h}} [f_1(\omega(t_1 - h))], \omega(r_1) \in A_1\right] \\
&= \mathbb{E}^x \left[\mathbb{E}^{B_{s+h}} [f_1(\omega(t_1 - h))], A\right],
\end{aligned} \tag{1.13}$$

where we used that

$$p_{t_1+t_2}(u, z) = \int_{\mathbb{R}} p_{t_1}(u, v) p_{t_2}(v, z)\, dv$$

for $t_1, t_2 \geq 0$ and $u, z \in \mathbb{R}$ at $(*)$ and Fubini's theorem at $(**)$. This also holds for general $k, n \in \mathbb{N}$, which can be written in a clearer way as

$$\mathbb{E}^x [X \circ \theta_s, A] = \mathbb{E}^x [\phi(B_{s+h}, h), A], \tag{1.14}$$

where

$$\phi(y, h) := \mathbb{E}^y \prod_{m=1}^{n} f_m(\omega(t_m - h)), \qquad y \in \mathbb{R},\ h > 0.$$

Writing $\phi(y,h)$ out explicitly in terms of the normal density as we have done at (1.8) in the proof of Proposition 1.9, we note that ϕ is jointly continuous in $(y,h) \in \mathbb{R} \times [0,t_1)$.

Next, we generalize (1.12) for all sets $A \in \mathcal{F}^0_{s+\frac{h}{2}}$ by applying the π-λ-theorem, Theorem A.7, to

$$\mathcal{P} := \left\{ A \in \mathcal{F}^0_{s+\frac{h}{2}} : A \text{ finite-dimensional} \right\}$$

and

$$\mathcal{L} := \left\{ A \in \mathcal{F}^0_{s+\frac{h}{2}} : (1.12) \text{ holds for } A \right\}.$$

$\mathcal{L}$ is a λ-system, since:

(i) $\Omega \in \mathcal{L}$, by doing similar calculation steps as in (1.13).

(ii) If $E, F \in \mathcal{L}$ and $E \subset F$, then $F \backslash E \in \mathcal{L}$ follows by noting that $\mathbb{1}_{F\backslash E} = \mathbb{1}_F - \mathbb{1}_E$.

(iii) Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ and $A_n \uparrow A$. $A \in \mathcal{L}$ follows by noting that $\mathbb{1}_A = \sum_{n\in\mathbb{N}} \mathbb{1}_{A_{n+1}\backslash A_n}$, the boundedness of X and (ii).

Since $\mathcal{P} \subset \mathcal{L}$ by (1.12), the π-λ-theorem yields that (1.12) also holds for each $A \in \sigma(\mathcal{P}) = \mathcal{F}^0_{s+\frac{h}{2}}$. Note that we had to take $s + \frac{h}{2}$, since the last point in time r_k of a finite-dimensional set has to be strictly smaller than $s+h$. Furthermore note that since $\mathcal{F}_s \subset \mathcal{F}^0_{s+\frac{h}{2}}$, (1.12) holds for every $A \in \mathcal{F}_s$. Using the joint continuity of ϕ, $\phi(y,0) = \mathbb{E}^y X$ for every $y \in \mathbb{R}$ and the continuity of Brownian paths, we get (1.11) by letting $h \downarrow 0$ in (1.14).

The monotone class theorem again extends the result to bounded random variables X. For this, let $\mathcal{P}$ be the π-system containing all finite-dimensional sets and set

$$\mathcal{H} := \{X \text{ bounded} : (1.11) \text{ holds for } A \in \mathcal{F}_s\}.$$

$\mathcal{H}$ is obviously a vector space satisfying property (ii) of the monotone class theorem by the bounded convergence theorem. Property (i) was already shown above. Hence, the proof is complete. ∎

1.2.1 Applications

There are some interesting and useful consequences following from the Markov property.

Proposition 1.11. *(a) Let X be a bounded random variable. Then for every $x \in \mathbb{R}$ and $s \geq 0$, it holds*

$$\mathbb{E}^x [X \mid \mathcal{F}_s] = \mathbb{E}^x \left[X \,\middle|\, \mathcal{F}_s^0\right] \quad \mathbb{P}^x\text{-a.s.} \tag{1.15}$$

(b) If $A \in \mathcal{F}_0$, then $\mathbb{P}^x(A) = 0$ or 1 for every $x \in \mathbb{R}$.

Remark. (b) is known as *Blumenthal's 0-1 law*. It is important to note that the decision whether $\mathbb{P}^x(A)$ is 0 or 1 in general depends on x.

Proof of Proposition 1.11. (a): Let $x \in \mathbb{R}$ and $s \geq 0$. First, we take special X as in (1.3). We will compute the left side of (1.15) explicitly using the Markov property, Theorem 1.10, and note that the result is also $\mathcal{F}_s^0$-measurable, which gives (1.15) for special X. For this, write

$$X(\omega) = X_1(\omega)(X_2 \circ \theta_s)(\omega),$$

where

$$X_1(\omega) = \prod_{m:t_m \leq s} f_m(\omega(t_m)) \quad \text{and} \quad X_2(\omega) = \prod_{m:t_m > s} f_m(\omega(t_m - s)).$$

Since X_1 is $\mathcal{F}_s^0 \subset \mathcal{F}_s$-measurable, the Markov property gives

$$\mathbb{E}^x [X \mid \mathcal{F}_s] = X_1 \mathbb{E}^x [X_2 \circ \theta_s \mid \mathcal{F}_s] = X_1 \mathbb{E}^{B_s} X_2 \quad \mathbb{P}^x\text{-a.s.}$$

The second factor of the right side is indeed also $\mathcal{F}_s^0$-measurable by Proposition 1.9.

We again apply the monotone class theorem to the π-system $\mathcal{P}$ containing all finite-dimensional sets and to

$$\mathcal{H} := \{X \text{ bounded} : (1.15) \text{ holds}\}.$$

$\mathcal{H}$ is a vector space and satisfies property (ii) of the monotone class theorem by using the dominated convergence theorem for conditional expectations. Property (i) has been proved above.

(b): Take $A \in \mathcal{F}_0$ and $x \in \mathbb{R}$. Using (a), we get

$$\mathbb{1}_A = \mathbb{E}^x [\mathbb{1}_A \mid \mathcal{F}_0] = \mathbb{E}^x \left[\mathbb{1}_A \,\middle|\, \mathcal{F}_0^0\right],$$

which means that $\mathbb{1}_A$ is $\mathcal{F}_0^0$-measurable. Since B_0 is constant $\mathbb{P}^x$-a.s., $\mathcal{F}_0^0$ consists only of events with $\mathbb{P}^x$-probability 0 or 1 implying that each $\mathcal{F}_0^0$-measurable random variable, and thus $\mathbb{1}_A$, is constant $\mathbb{P}^x$-a.s. Note that $\mathbb{1}_A$ can only have values 0 or 1. It follows

$$\mathbb{P}^x(A) = \mathbb{E}^x \mathbb{1}_A = 0 \text{ or } 1.$$

■

Corollary 1.12. *Let* $\tau_a := \inf\{t > 0 : B_t > 0\}$ *and* $\tau_b := \inf\{t > 0 : B_t = 0\}$. *Then:*

(a) $\mathbb{P}^0(\tau_a = 0) = 1$

(b) $\mathbb{P}^0(\tau_b = 0) = 1$

Proof. (a): Note that $\{\tau_a = 0\} \in \mathcal{F}_0$ (but $\notin \mathcal{F}_0^0$). Indeed, since τ_a is a stopping time by Proposition 1.5, we get

$$\{\tau_a = 0\} = \bigcap_{n\in\mathbb{N}} \left\{\tau_a \leq \frac{1}{n}\right\} \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\frac{1}{n}} = \mathcal{F}_0.$$

Therefore, we can use Blumenthal's 0-1 law, Proposition 1.11 (b), i.e. it suffices to show that the probability is strictly positive. For this, take $t > 0$ and note that

$$\mathbb{P}^0(\tau_a \leq t) \geq \mathbb{P}^0(B_t > 0) = \frac{1}{2}.$$

Since

$$\{\tau_a = 0\} = \bigcap_{n\in\mathbb{N}} \{\tau_a \leq t_n\}$$

for any decreasing sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \xrightarrow{n\to\infty} 0$, we get

$$\mathbb{P}^0(\tau_a = 0) = \lim_{t\downarrow 0} \mathbb{P}^0(\tau_a \leq t) \geq \frac{1}{2} > 0.$$

(b): Recognize that also the corresponding statement for $B_t > 0$ instead of $B_t < 0$ holds, i.e. $\mathbb{P}^0(\tau_{\tilde{a}} = 0) = 1$ for $\tau_{\tilde{a}} := \inf\{t > 0 : B_t < 0\}$. By path continuity of Brownian motion, it follows

$$\mathbb{P}^0(\tau_b = 0) \geq \mathbb{P}^0(\tau_a = 0 \wedge \tau_{\tilde{a}} = 0) = 1.$$

■

In contrast to Blumenthal's 0-1-law, another 0-1-law can guarantee the independence of x. It is about so-called *tail events* coming from the so-called *tail σ-algebra*

$$\mathcal{T} := \bigcap_{t>0} \mathcal{F}_t^*,$$

where $\mathcal{F}_t^*$ denotes the smallest σ-algebra with respect to which the projection π_s, defined in (1.1), is measurable for all $s \geq t$.

Proposition 1.13. *If $A \in \mathcal{T}$, then $\mathbb{P}^x(A) = 0$ for all $x \in \mathbb{R}$ or $\mathbb{P}^x(A) = 1$ for all $x \in \mathbb{R}$. Furthermore, if X is a $\mathcal{T}$-measurable random variable, then there exists a constant $c \in \mathbb{R}$, independent of x, such that $\mathbb{P}^x(X = c) = 1$ for all $x \in \mathbb{R}$.*

Proof of Proposition 1.13. Take $A \in \mathcal{T}$. Recall that

$$\tilde{B}_t(\omega) := \begin{cases} tB_{1/t}(\omega) = t\,\omega\left(\frac{1}{t}\right) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

is a standard Brownian motion (i.e. starting in 0). Since this transformation is bijective, we get that $\mathbb{P}^0(A) = 0$ or 1 by Blumenthal's 0-1 law. Let $x \in \mathbb{R}$ and note that the mapping $\omega(\cdot) \mapsto \omega(\cdot) + x$ transforms $\mathbb{P}^0$ into $\mathbb{P}^x$, but lets $\mathcal{T}$ unchanged. Therefore, $\mathbb{P}^x(A) = 0$ or 1. Now, we have to show that this probability does not depend on x. We do this by showing that $\mathbb{P}^x(A)$ is continuous in x, which implies the independence, since $\mathbb{P}^x(A)$ takes only values 0 or 1, and thus must be identically 0 or identically 1. Take some $s > 0$. Since $A \in \mathcal{F}_s^*$, there is some set $D \in \mathcal{F}$ such that $\mathbb{1}_A = \mathbb{1}_D \circ \theta_s$. The Markov property gives

$$\begin{aligned} \mathbb{P}^x(A) &= \mathbb{E}^x \mathbb{1}_A = \mathbb{E}^x\left[\mathbb{1}_D \circ \theta_s\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[\mathbb{1}_D \circ \theta_s \,|\, \mathcal{F}_s\right]\right] \\ &= \mathbb{E}^x\left[\mathbb{E}^{B_s}\mathbb{1}_D\right] = \int_{\mathbb{R}} p_s(x,y)\mathbb{P}^y(D)\, dy. \end{aligned}$$

The right side above is continuous in x, so $\mathbb{P}^x(A)$ is also.

Now, let X be a $\mathcal{T}$-measurable random variable. Since all events of the form $\{X \le y\}$ for $y \in \mathbb{R}$ have $\mathbb{P}^x$-probability 0 for all $x \in \mathbb{R}$ or $\mathbb{P}^x$-probability 1 for all $x \in \mathbb{R}$, there has to be a constant $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\mathbb{P}^x(X \le y) = 0$ for $y < c$ and $\mathbb{P}^x(X \le y) = 1$ for $y \ge c$. It follows that $X = c$ $\mathbb{P}^x$-a.s. for each $x \in \mathbb{R}$ implying that c is independent of x. ■

The next proposition is a special case of the well-known Itô formula from stochastic calculus and will be applied several times in Chapter 2. It is sometimes also referred to as *Dynkin's lemma.*

Proposition 1.14. *Let h be a C^2-function with compact support in $\mathbb{R}$. Then, for every $x \in \mathbb{R}$*

$$\mathbb{E}^x h(B_t) = h(x) + \mathbb{E}^x \int_0^t \frac{1}{2}\Delta h(B_s)\, ds \quad \text{for all } t \ge 0. \tag{1.16}$$

Consequently, the process

$$M_t := h(B_t) - \int_0^t \frac{1}{2}\Delta h(B_s)\, ds, \qquad t \ge 0$$

is a martingale with respect to $(\mathcal{F}_t)_{t\ge0}$ under $\mathbb{P}^x$ for every $x \in \mathbb{R}$.

Proof. The first part of the proof is based on the observation that the density of the $\mathcal{N}(x,t)$-distribution,

$$p_t(x,y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}},$$

where $t > 0$ and $x, y \in \mathbb{R}$, satisfies the heat equation

$$\frac{\partial}{\partial t} p_t(x,y) = \Delta_y p_t(x,y) \tag{1.17}$$

for all $t > 0$, and $x, y \in \mathbb{R}$, which can be seen by a direct calculation. Here, Δ_y denotes the Laplacian with respect to y. We need to pay attention on the fact that the partial derivatives of $p_t(x,y)$ blow up at $t = 0$ and $x = y$. But for any $\epsilon > 0$ all the partial derivatives (both $\frac{\partial}{\partial t}$ and Δ_y) are bounded on the region $[\epsilon, \infty) \times \mathbb{R}^2$. This and the fundamental theorem of calculus give for $0 < \epsilon < t$ and $x \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}^x h(B_t) &= \int_{\mathbb{R}} h(y) p_t(x,y)\, dy \\ &= \underbrace{\int_{\mathbb{R}} \int_{\epsilon}^{t} h(y) \frac{\partial}{\partial s} p_s(x,y)\, ds\, dy}_{=:A_1} + \underbrace{\int_{\mathbb{R}} h(y) p_\epsilon(x,y)\, dy}_{=\mathbb{E}^x h(B\epsilon)}. \end{aligned} \tag{1.18}$$

In a second step, by (1.17) and Fubini's theorem, we get for A_1

$$\begin{aligned} A_1 &= \int_{\mathbb{R}} \int_{\epsilon}^{t} \frac{1}{2} h(y) \Delta_y p_s(x,y)\, ds\, dy \\ &= \int_{\epsilon}^{t} \underbrace{\int_{\mathbb{R}} \frac{1}{2} h(y) \Delta_y p_s(x,y)\, dy}_{=:A_2}\, ds. \end{aligned}$$

Now, we can integrate by parts twice for A_2 to get the Laplacian from $p_s(x,y)$ to $h(y)$. Note that the boundary terms vanish, since h has compact support. This gives

$$A_2 = -\int_{\mathbb{R}} \frac{1}{2} \frac{\partial}{\partial y} h(y) \frac{\partial}{\partial y} p_s(x,y)\, dy = \int_{\mathbb{R}} \frac{1}{2} \Delta h(y) p_s(x,y)\, dy$$

implying

$$A_1 = \int_{\mathbb{R}} \int_{\epsilon}^{t} \frac{1}{2} \Delta h(y) p_s(x,y)\, ds\, dy = \mathbb{E}^x \int_{\epsilon}^{t} \frac{1}{2} \Delta h(B_s)\, ds,$$

where we used Fubini's theorem again. Finally, (1.18) becomes

$$\mathbb{E}^x h(B_t) = \mathbb{E}^x h(B_\epsilon) + \mathbb{E}^x \int_{\epsilon}^{t} \frac{1}{2} \Delta h(B_s)\, ds.$$

Since h and Δh are bounded and continuous, the bounded convergence theorem yields (1.16) for $\epsilon \to 0$.

For the second part, note that (1.16) can be written as

$$\mathbb{E}^x M_t = h(x) \tag{1.19}$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. $(M_t)_{t\geq 0}$ is obviously adapted to $(\mathcal{F}_t)_{t\geq 0}$. M_t is also integrable with respect to $\mathbb{P}^x$ for every $t \geq 0$ by (1.19). The martingale property follows from the Markov property. Indeed, let $0 \leq s < t$ and calculate

$$\begin{aligned}
\mathbb{E}^x\left[M_t \mid \mathcal{F}_s\right] &= \mathbb{E}^x\left[h(B_{t-s})\circ\theta_s + \int_0^s \frac{1}{2}\Delta h(B_u)\,du + \int_0^{t-s}\frac{1}{2}\Delta h(B_u)\circ\theta_s\,du \,\middle|\, \mathcal{F}_s\right] \\
&= \mathbb{E}^{B_s}h(B_{t-s}) + \int_0^s \frac{1}{2}\Delta h(B_u)\,du + \mathbb{E}^{B_s}\int_0^{t-s}\frac{1}{2}\Delta h(B_u)\,du \\
&= \mathbb{E}^{B_s}M_{t-s} + \int_0^s \frac{1}{2}\Delta h(B_u)\,du \\
&\overset{(*)}{=} h(B_s) + \int_0^s \frac{1}{2}\Delta h(B_u)\,du \\
&= M_s \quad \mathbb{P}^x\text{-a.s.},
\end{aligned}$$

where we used (1.19) at $(*)$. ■

Remark. (a) Note that this proof can be done in a similar way for Brownian motion and h in n dimensions.
(b) This result can be generalized to bounded functions, since every bounded function is the limit of a sequence of functions with compact support.

1.3 The strong Markov property

In this section, it is about to prove the strong Markov property. The big difference to the former Markov property is that it takes stopping times into account instead of only fixed points in time. Therefore, we first state a definition that allows us to deal with "stopping time σ-algebras". Afterwards, we will show some basic results for it required by the proof of the strong Markov property.

Definition 1.15. Let τ be a stopping time. Define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Proposition 1.16. *Let τ be a stopping time. Then:*

(a) $\mathcal{F}_\tau$ *is a σ-algebra.*

(b) $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$

Proof. Let $t \geq 0$ always be arbitrary.

(a): We show that $\mathcal{F}_\tau$ satisfies the properties of a σ-algebra:

(i) $\emptyset \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t$. So, $\emptyset \in \mathcal{F}_\tau$.

(ii) For $A \in \mathcal{F}_\tau$,

$$A^c \cap \{\tau \leq t\} = \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \setminus \underbrace{(A \cap \{\tau \leq t\})}_{\in \mathcal{F}_t}.$$

So, $A^c \in \mathcal{F}_\tau$.

(iii) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets with $A_n \in \mathcal{F}_\tau$ for each $n \in \mathbb{N}$. Then

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap \{\tau \leq t\} = \bigcup_{n\in\mathbb{N}} \underbrace{(A_n \cap \{\tau \leq t\})}_{\in \mathcal{F}_t} \in \mathcal{F}_t$$

implying $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}_\tau$.

(b): Let $\tilde{\mathcal{F}}_\tau$ be the right side of the statement. For the containment $\subseteq$, let $A \in \mathcal{F}_\tau$. We get

$$\begin{aligned} A \cap \{\tau < t\} &= A \cap \bigcup_{n\in\mathbb{N}} \left\{\tau \leq t - \frac{1}{n}\right\} \\ &= \bigcup_{n\in\mathbb{N}} \underbrace{\left(A \cap \left\{\tau \leq t - \frac{1}{n}\right\}\right)}_{\in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t} \in \mathcal{F}_t \end{aligned}$$

implying $A \in \tilde{\mathcal{F}}_\tau$. For the converse containment, take $\tilde{A} \in \tilde{\mathcal{F}}_\tau$. Here, we get

$$\begin{aligned} \tilde{A} \cap \{\tau \leq t\} &= \tilde{A} \cap \bigcap_{n\in\mathbb{N}} \left\{\tau < t + \frac{1}{n}\right\} \\ &= \bigcap_{n\in\mathbb{N}} \left(\tilde{A} \cap \left\{\tau < t + \frac{1}{n}\right\}\right) \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t. \end{aligned}$$

Hence, $\tilde{A} \in \mathcal{F}_\tau$. ∎

Lemma 1.17. *Let all τ's appearing below be stopping times. Then:*

(a) τ is $\mathcal{F}_\tau$-measurable.

(b) $\tau_1 \leq \tau_2$ implies $\mathcal{F}_{\tau_1} \subseteq \mathcal{F}_{\tau_2}$.

(c) If $\tau_n \downarrow \tau$, then $\mathcal{F}_\tau = \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}$.

(d) If the process $X = (X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and has right-continuous paths, then $X_\tau \mathbb{1}_{\{\tau<\infty\}}$ is $\mathcal{F}_\tau$-measurable.

Proof. (a): It suffices to show that $\{\tau \leq s\} = \{\tau \in [0, s]\} \in \mathcal{F}_\tau$ for $s \geq 0$, since it holds

$$\{\tau \in (r, s]\} = \{\tau \leq s\} \cap \underbrace{\{\tau > r\}}_{\{\tau\leq r\}^c}$$

for each $r, s \geq 0$ with $r < s$ and since half-open intervals generate $\mathcal{B}(\mathbb{R})$. So, let $s, t \geq 0$ and get

$$\{\tau \leq s\} \cap \{\tau \leq t\} = \underbrace{\{\tau \leq s \wedge t\}}_{\in\mathcal{F}_{s\wedge t}\subset\mathcal{F}_t} \in \mathcal{F}_t$$

implying $\{\tau \leq s\} \in \mathcal{F}_\tau$.

(b): Take $A \in \mathcal{F}_{\tau_1}$ and note that $\{\tau_2 \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\}$ since $\{\tau_2 \leq t\} \subseteq \{\tau_1 \leq t\}$ for each $t \geq 0$. It follows for $t \geq 0$

$$A \cap \{\tau_2 \leq t\} = \underbrace{A \cap \{\tau_1 \leq t\}}_{\in\mathcal{F}_t} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t.$$

Hence, $A \in \mathcal{F}_{\tau_2}$.

(c): Since $\tau \leq \tau_n$ for every $n \in \mathbb{N}$, the containment $\subseteq$ follows by (b). For the converse containment, take $A \in \bigcap_{n\in\mathbb{N}} \mathcal{F}_{\tau_n}$ and $t \geq 0$, and see that

$$\begin{aligned} A \cap \{\tau < t\} &= A \cap \bigcup_{n\in\mathbb{N}} \{\tau_n < t\} \\ &= \bigcup_{n\in\mathbb{N}} \underbrace{(A \cap \{\tau_n < t\})}_{\in\mathcal{F}_t} \in \mathcal{F}_t \end{aligned}$$

implying $A \in \mathcal{F}_\tau$.

(d): For a general stopping time τ, note that

$$X_\tau \mathbb{1}_{\{\tau<\infty\}} = \lim_{n\to\infty} X_{\tau\wedge n} \mathbb{1}_{\{\tau<\infty\}}.$$

Thus, since τ is $\mathcal{F}_\tau$-measurable by (a) and $X_{\tau\wedge n}$ is (also) $\mathcal{F}_{\tau\wedge n} \subseteq \mathcal{F}_\tau$-measurable for each $n \in \mathbb{N}$, it suffices to show that $X_{\tilde\tau}$ is $\mathcal{F}_{\tilde\tau}$-measurable for finite stopping times $\tilde\tau$.

Assume first that the stopping time τ takes only finitely many values $t_1, t_2, \ldots$. For $a \in \mathbb{R}$ and $t \geq 0$, we can write

$$\{X_\tau \leq a\} \cap \{\tau \leq t\} = \bigcup_{\substack{k\in\mathbb{N}\\ t_k \leq t}} \underbrace{\{\tau = t_k, X_{t_k} \leq a\}}_{\in \mathcal{F}_{t_k} \subseteq \mathcal{F}_t} \in \mathcal{F}_t,$$

since τ is a discrete stopping time and since X is adapted to $(\mathcal{F}_t)_{t\geq 0}$. This implies that $\{X_\tau \leq a\} \in \mathcal{F}_\tau$, and therefore X_τ is $\mathcal{F}_\tau$-measurable (see (a)).

Now, let τ be a finite stopping time, i.e. $\tau < \infty$. We approximate τ by a sequence of decreasing discrete stopping times $(\tau_n)_{n\in\mathbb{N}}$ as follows:

$$\tau_n := \frac{k+1}{2^n}, \qquad \text{if } \frac{k}{2^n} \leq \tau < \frac{k+1}{2^n} \quad \text{for some } k \in \mathbb{N}_0. \tag{1.20}$$

Note that $\tau_n \downarrow \tau$ and furthermore that τ_n is a stopping time for each $n \in \mathbb{N}$. Indeed, for $t \geq 0$, we find some $k \in \mathbb{N}_0$ such that $\frac{k}{2^n} \leq t < \frac{k+1}{2^n}$. Then

$$\{\tau_n \leq t\} = \left\{\tau < \frac{k}{2^n}\right\} \in \mathcal{F}_{k/2^n} \subset \mathcal{F}_t.$$

Since X has right-continuous paths, $X_{\tau_n} \xrightarrow{n\to\infty} X_\tau$. For $n \geq m$, X_{τ_n} is $\mathcal{F}_{\tau_n}$-measurable, since τ_n is discrete, and thus is $\mathcal{F}_{\tau_m}$-measurable by (b), since $\tau_n \leq \tau_m$. It follows that X_τ is $\mathcal{F}_{\tau_m}$-measurable for each $m \in \mathbb{N}$ implying that X_τ is $\mathcal{F}_\tau$-measurable by (c), which completes the proof. ■

Now, we are ready to state and prove the strong Markov property.

Theorem 1.18 (Strong Markov property). *Let X be a bounded random variable and τ be a stopping time. Then for every $x \in \mathbb{R}$, it holds*

$$\mathbb{E}^x\left[X \circ \theta_\tau \,|\, \mathcal{F}_\tau\right] = \mathbb{E}^{B_\tau} X \qquad \mathbb{P}^x\text{-a.s. on } \{\tau < \infty\}. \tag{1.21}$$

Proof. Let $x \in D$ throughout the whole proof. Again, the plan is to prove (1.21) first for discrete stopping times taking only countable many values. Afterwards, we approximate arbitrary stopping times by discrete ones from above and take X of special form such that we can finally use the monotone class theorem.

So, assume first that τ takes only countable many values $t_1, t_2, \ldots$ and ∞. It is important to note that the right side of (1.21) is $\mathcal{F}_\tau$-measurable by Proposition 1.9

and Lemma 1.17 (d). To verify the defining property of the conditional expectation in (1.21), we need to show that

$$\mathbb{E}^x\left[X \circ \theta_\tau, A\right] = \mathbb{E}^x\left[\mathbb{E}^{B_\tau} X, A\right] \tag{1.22}$$

for $A \in \mathcal{F}_\tau$ with $A \subset \{\tau < \infty\}$. For this, write

$$\begin{aligned}
\mathbb{E}^x\left[X \circ \theta_\tau, A\right] &= \mathbb{E}^x\Big[(X \circ \theta_\tau) \cdot \underbrace{\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau = t_n\}}}_{=1}, A\Big] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}^x\left[X \circ \theta_\tau, A \cap \{\tau = t_n\}\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}^x\left[X \circ \theta_{t_n}, A \cap \{\tau = t_n\}\right] \\
&\overset{(*)}{=} \sum_{n\in\mathbb{N}} \mathbb{E}^x\left[\mathbb{E}^{B_{t_n}} X, A \cap \{\tau = t_n\}\right] \\
&= \sum_{n\in\mathbb{N}} \mathbb{E}^x\left[\mathbb{E}^{B_\tau} X, A \cap \{\tau = t_n\}\right] \\
&= \mathbb{E}^x\left[\mathbb{E}^{B_\tau} X, A\right],
\end{aligned}$$

where we used the Markov property, Theorem 1.10, at $(*)$ recognizing that it applied, since $A \cap \{\tau = t_n\} \in \mathcal{F}_{t_n}$, $n \in \mathbb{N}$, by the definition of $\mathcal{F}_\tau$.

If τ is an arbitrary stopping time, we approximate it from above by a sequence of stopping times $(\tau_k)_{k\in\mathbb{N}}$ as we did in (1.20). Let $\tau_k = \infty$ for all $k \in \mathbb{N}$, if $\tau = \infty$. Let X be special — see (1.3) — and take $A \in \mathcal{F}_\tau \subset \mathcal{F}_{\tau_k}$, $k \in \mathbb{N}$, such that $A \subset \{\tau < \infty\}$. The first part above gives for $k \in \mathbb{N}$

$$\mathbb{E}^x\left[X \circ \theta_{\tau_k}, A\right] = \mathbb{E}^x\left[\mathbb{E}^{B_{\tau_k}} X, A\right].$$

Now, we have to pass to the limit as $k \to \infty$. For the left side, we write

$$(X \circ \theta_{\tau_k})(\omega) = \prod_{m=1}^{n} f_m(\omega(t_m + \tau_k)) \xrightarrow{k\to\infty} \prod_{m=1}^{n} f_m(\omega(t_m + \tau)) = (X \circ \theta_\tau)(\omega),$$

where we used the (right) continuity of Brownian paths. For the right side, recall from the proof of Proposition 1.9 that the function $y \mapsto \mathbb{E}^y X$ is continuous, which finally gives (1.22)

Finally, applying the monotone class theorem to the π-system $\mathcal{P}$ containing all finite-dimensional sets and to

$$\mathcal{H} := \{X \text{ bounded} : (1.22) \text{ holds for } A \in \mathcal{F}_\tau \text{ with } A \subset \{\tau < \infty\}\}$$

gives the general case for bounded random variables X. Obviously, $\mathcal{H}$ is a vector space and by using the bounded convergence theorem, we can see that $\mathcal{H}$ satisfies property (ii) of the monotone class theorem. The special case above shows property (i). ■

Chapter 2

Brownian Motion and the Dirichlet Problem

This chapter contains the main results of the thesis and treats the connection of n-dimensional Brownian motion and the Dirichlet problem.

One of the most important and interesting partial differential equations, which is a topic of every PDE course, is the Laplace equation:

$$\Delta u = 0 \quad \text{on } \mathbb{R}^n, \tag{2.1}$$

where Δ denotes the Laplace operator defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

A function (on $\mathbb{R}^n$) which satisfies (2.1) is called *harmonic*.

It describes the equilibrium temperature distribution, in which one is interested especially for a given body D with fixed temperature on the boundary ∂D. This means, we want to find functions h on D satisfying $\Delta h = 0$ on D and $h = f$ on ∂D for prescribed boundary values f. This problem is called the *Dirichlet problem on D*. The bodies D will be mathematically described throughout this chapter by domains, i.e. open, connected subsets of $\mathbb{R}^n$.

In the 1940's and 1950's, it was recognized by the pioneers Kakutani, Kac and Doob that there is a deep connection between n-dimensional Brownian motion, harmonic functions and the Dirichlet problem. As a consequence of this probabilistic approach, we can approximate solutions to the Dirichlet problem by Monte-Carlo simulations while requiring less smoothness of the boundary.

In a first step (Section 2.1), we are going to treat some basic results about harmonic functions, which we will need from time to time later on. In Section 2.2 we state and prove some first results about the connection between Brownian motion and harmonic functions and prepare for Section 2.3, in which we will care about the Dirichlet problem on bounded and unbounded domains. Finally, in

Section 2.4, we see how to solve a boundary value problem on bounded domains involving the Poisson equation in a probabilistic way.

2.1 Harmonic Functions

We will see that harmonicity can be defined in two ways. Our first goal is to prove their equivalence. Before we start, let us clarify some notation. The closed ball in $\mathbb{R}^n$ is denoted by $B(x,r)$ and its boundary by $\partial B(x,r)$. The definition is

$$B(x,r) := \{y \in \mathbb{R}^n : |x-y| \le r\}\,, \quad \partial B(x,r) := \{y \in \mathbb{R}^n : |x-y| = r\}\,,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

The normalized surface measure of total mass 1 on $\partial B(x,r)$ is called $\sigma_{x,r}(dy)$, the Lebesgue measure on $B(x,r)$ simply dy. The volume of $B(0,1)$ in n dimensions is denoted by V_n, so the volume of $B(x,r)$ is $V_n r^n$. The surface of $B(x,r)$ in n dimensions is $nV_n r^{n-1}$.

Definition 2.1 (Mean value property)**.** A continuous function h on D satisfies the *mean value property* if

$$h(x) = \int_{\partial B(x,r)} h(y)\, \sigma_{x,r}(dy) \tag{2.2}$$

for all $x \in D$ and $r > 0$ such that $B(x,r) \subset D$.

Theorem 2.2. *Let h be a function on D. Then the following statements are equivalent:*

(a) h is continuous and satisfies the mean value property.

(b) $h \in C^2(D)$ and satisfies $\Delta h = 0$ on D.

Proof. Suppose (a) holds. First we want to prove that $h \in C^\infty(D)$. For this, fix $x \in D$ and take a radial C^∞-function ϕ with support in $[0,\epsilon]$ for $\epsilon > 0$ such that $B(x,\epsilon) \subset D$. A change to spherical coordinates (Theorem A.3) gives

$$\begin{aligned}\int_{\mathbb{R}^n} \phi\,(|y-x|)\, h(y)\, dy &= \int_{B(0,\epsilon)} \phi\,(|u|)\, h(x+u)\, du \\ &= nV_n \int_0^\epsilon \phi(r) r^{n-1} \left(\int_{\partial B(x,r)} h(y)\, \sigma_{x,r}(dy)\right) dr \\ &= h(x) nV_n \int_0^\epsilon \phi(r) r^{n-1}\, dr,\end{aligned} \tag{2.3}$$

where we used (2.2) in the last step. The left side of (2.3) is a C^∞-function since all derivatives can be put under the integral (ϕ has support in $[0,\epsilon]$). If the integral on the right side of (2.3) is nonzero, it follows that $h \in C^\infty(D)$.

For the rest, it suffices to show that for $h \in C^2(D)$ the mean value property is equivalent to $\Delta h = 0$ on D. We will prove this analytically using the Gauss-Green formula (Theorem A.2). Again for $x \in D$ and $r > 0$ such that $B(x,r) \subset D$ this gives

$$\begin{aligned}\int_{B(x,r)} \Delta h(y)\,dy &= nV_n r^{n-1} \int_{\partial B(x,r)} \nabla h(y)\cdot \nu\,\sigma_{x,r}(dy)\\ &= nV_n r^{n-1} \int_{\partial B(x,r)} \nabla h(y)\cdot \frac{y-x}{r}\,\sigma_{x,r}(dy)\\ &= nV_n r^{n-1} \int_{\partial B(0,1)} \nabla h(x+ry)\cdot y\,\sigma_{0,1}(dy)\\ &= nV_n r^{n-1} \frac{d}{dr}\int_{\partial B(0,1)} h(x+ry)\,\sigma_{0,1}(dy)\\ &= nV_n r^{n-1} \frac{d}{dr}\int_{\partial B(x,r)} h(y)\,\sigma_{x,r}(dy),\end{aligned} \tag{2.4}$$

where ν is the outward normal vector on $\partial B(x,r)$. Suppose $\Delta h = 0$ on D. (2.4) gives that

$$\int_{\partial B(x,r)} h(y)\,\sigma_{x,r}(dy)$$

is constant in r. Using Lemma A.4, it follows

$$\int_{\partial B(x,r)} h(y)\,\sigma_{x,r}(dy) = \lim_{t\to 0}\int_{\partial B(x,t)} h(y)\,\sigma_{x,t}(dy) = h(x),$$

which proves the mean value property. Now assume that (2.2) holds for any ball $B(x,r) \subset D$. By (2.4), it follows

$$\int_{B(x,r)} \Delta h(y)\,dy = 0. \tag{2.5}$$

It needs to be shown that $\Delta h \equiv 0$ on D. Recall that Δh is continuous (since h was proven to be $C^2(D)$ in the first step). This means that if $\Delta h \not\equiv 0$ on D, there exist $x_0 \in D$ and $r_0 > 0$ with $B(x_0,r_0) \subset D$ such that, say w.l.o.g., $\Delta h > 0$ within $B(x_0,r_0)$. Using (2.5), we get a contradiction:

$$0 = \int_{B(x_0,r_0)} \Delta h(y)\,dy > 0$$

So, $\Delta h \equiv 0$ on D. ■

Definition 2.3 (Harmonic function)**.** A function h on D is called *harmonic* if it satisfies the equivalent properties from Theorem 2.2.

Here are some typical examples of harmonic functions in several dimensions, which we will need later on.

Example 2.4. Let

$$h(x) = \begin{cases} x & \text{if } n = 1, \\ \log|x| & \text{if } n = 2, \\ \frac{1}{|x|^{n-2}} & \text{if } n \geq 3. \end{cases}$$

Using criterion (b) in Theorem 2.2, we can show that h is harmonic in $\mathbb{R}$ if $n = 1$ and harmonic in $\mathbb{R}^n \setminus \{0\}$ if $n \geq 2$.

There is an interesting basic result contrasting Example 2.4.

Proposition 2.5. *Every nonnegative harmonic function on $\mathbb{R}^n$ is constant.*

Proof. Choose arbitrary $y,z \in \mathbb{R}^n$ and take $s > 0$. Define $r_s := s + |y - z|$ implying $B(y,s) \subset B(z, r_s)$. Since h is harmonic on $\mathbb{R}^n$, we can set $\phi = \mathbb{1}_{[0,r_s]}$ in (2.3) to get

$$h(z) = \frac{1}{V_n r_s^n} \int_{B(z,r_s)} h(x)\, dx.$$

Since h is nonnegative, it follows

$$h(z) \geq \frac{1}{V_n r_s^n} \int_{B(y,s)} h(x)\, dx = \frac{1}{V_n s^n} \int_{B(y,s)} h(x)\, dx \cdot \left(\frac{s}{r_s}\right)^n = h(y) \left(\frac{s}{r_s}\right)^n .$$

Letting $s \uparrow \infty$ gives $\frac{s}{r_s} \to 1$ implying $h(z) \geq h(y)$. Since $y,z \in \mathbb{R}^n$ were chosen arbitrarily, h is constant. ∎

The following "maximum principle" of the Laplace equation is a purely analytic result which will be used several times in this chapter.

Proposition 2.6 (Maximum Principle for Harmonic Functions)**.** *Let h be a harmonic function on D.*

(a) If h attains its maximum on D, then it is constant on D.

(b) If D is bounded and $h \in C(\overline{D})$, then

$$\max_{\overline{D}} h = \max_{\partial D} h.$$

Remark. Assertion (a) is called the *strong maximum principle* and (b) is the *maximum principle* for harmonic functions. Replace h by $-h$ to get similar assertions for the minimum case.

Proof of Proposition 2.6. First prove (a). Assume h attains its maximum $M := \max_D h$ at $x_0 \in D$ and consider the set

$$S := \{x \in D : h(x) = M\}.$$

We want to prove that S is open, relatively closed (i.e. closed in the relative topology of D) and nonempty. Since D is connected, this would imply $S = D$ (and thus $h \equiv M$ on D).
Openness: Take $x \in S$. Since h is harmonic on D, we can set $\phi = \mathbb{1}_{[0,r]}$ in (2.3) to get

$$M = h(x) = \frac{1}{V_n r^n} \int_{B(x,r)} h(y)\, dy \quad \text{if } B(x,r) \subset D. \tag{2.6}$$

It follows that $h \equiv M$ within $B(x,r)$, since M is the maximum and (2.6) would fail otherwise. So, $B(x,r) \subset S$.
Relative closedness: Take a sequence $(x_n)_{n\in\mathbb{N}}$ in S with $x_n \xrightarrow{n\to\infty} x$. We may assume that $x \in D$, since we want to show closedness in the relative topology of D. Since h is continuous, it holds $h(x_n) \xrightarrow{n\to\infty} h(x)$, implying $x \in S$.
Nonemptyness: Obviously, S is not empty since it was assumed $x_0 \in S$.

Now prove (b). If h is constant on $\overline{D}$, the assertion follows immediately. So, assume that h is not constant and, for a contradiction, h attains its maximum in D. But in this case, (a) implies that h has to be constant. So, h attains its maximum on the boundary ∂D. ∎

Corollary 2.7. *Suppose D is bounded and $h_1, h_2 \in C(\overline{D})$ are two harmonic functions. If $h_1 = h_2$ on ∂D, then $h_1 = h_2$ on D.*

Proof. Set $h := h_1 - h_2$, which is again harmonic on D, $\in C(\overline{D})$ and $\equiv 0$ on ∂D. Proposition 2.6 (b), both the maximum and the minimum case, gives $h \equiv 0$ on $\overline{D}$. The result follows. ∎

Remark. Looking ahead, note that Proposition 2.6 and its Corollary 2.7 say that if a solution to the Dirichlet problem exists, then it is unique (provided D is bounded and f is continuous).

2.2 Brownian motion comes into play

In this section, we will see how Brownian motion connects with harmonic functions and afterwards with the Dirichlet problem. But for this, we again will give a suitable mathematical context: a filtered probability space on which n-dimensional Brownian motion can live.

The probability space, which is again denoted by $(\Omega, \mathcal{F}, \mathbb{P})$ can be constructed from the one-dimensional case. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ denote the probability space for the one-dimensional case from Chapter 1. We define

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\tilde{\Omega}^n, \tilde{\mathcal{F}}^{\otimes n}, \tilde{\mathbb{P}}^{\otimes n}).$$

Hence, the n-dimensional process

$$B : [0, \infty) \times \Omega \to \mathbb{R}^n, (t, \omega) \mapsto (B_t^1(\omega_1), \dots, B_t^n(\omega_n)),$$

where all B^i, $i \in [n]$, are **independent** one-dimensional Brownian motions, will represent n-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

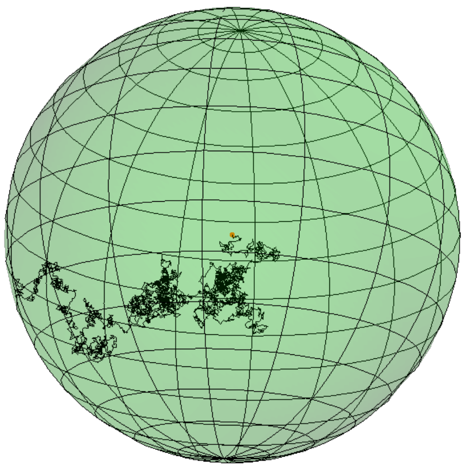

Figure 2.1: Three-dimensional Brownian motion started at the origin of a sphere and stopped at the boundary

The construction of the well-known family of probability measures $(\mathbb{P}^x)_{x \in \mathbb{R}^n}$ can be adapted from the one-dimensional case, see (1.2). The filtrations $(\mathcal{F}_t^0)_{t \geq 0}$ and $(\mathcal{F}_t)_{t \geq 0}$ are appropriately defined as

$$\mathcal{F}_t^0 := (\tilde{\mathcal{F}}_t^0)^{\otimes n} \quad \text{and} \quad \mathcal{F}_t := \tilde{\mathcal{F}}_t^{\otimes n}, \qquad t \geq 0,$$

where $(\tilde{\mathcal{F}}_t^0)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ denote the filtrations from the one-dimensional case.

We will need this concrete mathematical context later when we prove Blumenthal's 0-1 law for Brownian motion in n dimensions.

Note that our versions of the (strong) Markov property from Chapter 1, Theorem 1.10 and 1.18, also hold for Brownian motion in n dimensions. The proofs can be adapted.

Furthermore, recall that n-dimensional Brownian motion is rotationally invariant. In order to use this property in the sequel, we need the following result.

Lemma 2.8. *$\sigma_{x,r}$ is the unique rotationally invariant probability measure on $\partial B(x,r)$.*

Proof. Assume μ is a rotationally invariant probability measure on $\partial B(x,r)$ and let ϕ_μ be its characteristic function (i.e. $\phi_\mu(u) = \int_{\partial B(x,r)} e^{i\langle u,v\rangle}\mu(dv)$, $u \in \mathbb{R}^n$). Analogously, let ϕ_σ be the characteristic function of $\sigma_{x,r}$. Also the characteristic functions are rotationally invariant, since for a rotation matrix R, it holds for ϕ_μ (and ϕ_σ respectively)

$$\begin{aligned}\phi_\mu(Ru) &= \int_{\partial B(x,r)} e^{i\langle Ru,v\rangle}\mu(dv) = \int_{\partial B(x,r)} e^{i\langle u,R^Tv\rangle}\mu(dv)\\ &= \int_{\partial B(x,r)} e^{i\langle u,v\rangle}\mu(dv) = \phi_\mu(u).\end{aligned}$$

Hence, there exist two functions ϕ_μ^* and ϕ_σ^* depending only on $|u|$ such that

$$\phi_\mu(u) = \phi_\mu^*(|u|) \quad \text{and} \quad \phi_\sigma(u) = \phi_\sigma^*(|u|).$$

Thus, by the uniqueness theorem for characteristic functions (Theorem A.10), it suffices to show that $\phi_\mu^* = \phi_\sigma^*$. We compute

$$\begin{aligned}\phi_\mu^*(r) &= \int_{\partial B(x,r)} \phi_\mu^*(r)\,\sigma_{x,r}(du) = \int_{\partial B(x,r)} \phi_\mu(u)\,\sigma_{x,r}(du)\\ &= \int_{\partial B(x,r)} \left(\int_{\partial B(x,r)} e^{i\langle u,v\rangle}\,\mu(dv)\right)\sigma_{x,r}(du)\\ &\overset{(*)}{=} \int_{\partial B(x,r)} \left(\int_{\partial B(x,r)} e^{i\langle u,v\rangle}\,\sigma_{x,r}(du)\right)\mu(dv)\\ &= \int_{\partial B(x,r)} \phi_\sigma(v)\,\mu(dv) = \int_{\partial B(x,r)} \phi_\sigma^*(r)\,\mu(dv) = \phi_\sigma^*(r),\end{aligned}$$

using Fubini's theorem at $(*)$. ■

Theorem 2.9. *Assume h is a continuous function on $\mathbb{R}^n$ such that $\mathbb{E}^x|h(B_t)| < \infty$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Then the following statements are equivalent:*

(a) *h is harmonic.*

(b) *$h(x) = \mathbb{E}^x h(B_t)$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.*

Proof. Suppose (a) holds and let $t > 0$. Since h is harmonic, it satisfies the mean value property and we can rewrite (2.3) for $\phi(r) := \frac{1}{\sqrt{2\pi t}^{\frac{n}{2}}} e^{-\frac{r^2}{2t}}$ and $D := \mathbb{R}^n$ (i.e. the restriction on the support is not relevant) to get

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|y-x|^2}{2t}} h(y)\,dy &= h(x) n V_n \frac{1}{\sqrt{2\pi t}^{\frac{n}{2}}} \int_0^\infty e^{-\frac{r^2}{2t}} r^{n-1}\,dr \\ &= h(x) \underbrace{\frac{1}{\sqrt{2\pi t}^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{2t}}\,dy}_{=1} = h(x), \quad x \in \mathbb{R}^n.\end{aligned}$$

Note that B_t is normally distributed with mean x and covariance matrix tI under $\mathbb{P}^x$. Hence, it holds for $x \in \mathbb{R}^n$

$$h(x) = \frac{1}{\sqrt{2\pi t}^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|y-x|^2}{2t}} h(y)\,dy = \mathbb{E}^x h(B_t).$$

For $t = 0$, it holds $B_t = x$ $\mathbb{P}^x$-a.s., so it follows $\mathbb{E}^x h(B_t) = \mathbb{E}^x h(x) = h(x)$.

For the converse, assume (b) and prove the mean value property for h. Take $x \in \mathbb{R}^n$. Using the Markov property for Brownian motion, Theorem 1.10, we get that $(\mathbb{E}^x h(B_t))_{t \geq 0}$ is a martingale under $\mathbb{P}^x$: For $s, t \geq 0$, it holds

$$\mathbb{E}^x \left[h(B_{t+s}) \,|\, \mathcal{F}_s\right] = \mathbb{E}^{B_s} h(B_t) = h(B_s) \quad \mathbb{P}^x\text{-a.s.}$$

For some $r > 0$, let τ be the first exit time of $B(x, r)$. By the stopping time theorem, it follows for $t \geq 0$

$$h(x) = \mathbb{E}^x h(B_t) = \mathbb{E}^x \left[\mathbb{E}^x \left[h(B_t) \,|\, \mathcal{F}_{t\wedge\tau}\right]\right] = \mathbb{E}^x h(B_{t\wedge\tau}). \tag{2.7}$$

Note that the distribution of B_τ is rotationally invariant and so is $\sigma_{x,r}$ on $\partial B(x, r)$ by Lemma 2.8. To apply the bounded convergence theorem to (2.7), we show that $\mathbb{E}^x h(B_{t\wedge\tau})$ is bounded:

$$\begin{aligned}\mathbb{E}^x|h(B_{t\wedge\tau})| &\leq \mathbb{E}^x \left[|h(B_t)|\mathbb{1}_{\{\tau \geq t\}}\right] + \mathbb{E}^x \left[|h(B_\tau)|\mathbb{1}_{\{\tau < t\}}\right] \\ &\leq \mathbb{E}^x|h(B_t)| + \mathbb{E}^x|h(B_\tau)| < \infty,\end{aligned}$$

since $\mathbb{E}^x|h(B_t)| < \infty$ by assumption and $\mathbb{E}^x|h(B_\tau)| = \int_{\partial B(x,r)} |h(y)|\,\sigma_{x,r}(dy) < \infty$, where we used that h is continuous on $\mathbb{R}^n$. Eventually by the continuity of h, letting $t \to \infty$ in (2.7) gives $h(x) = \mathbb{E}^x h(B_\tau) = \int_{\partial B(x,r)} h(y)\,\sigma_{x,r}(dy)$. ∎

The following probabilistic proof demonstrates how to apply Theorem 2.9 to a basic result in PDE theory. The proof is done via "coupling" two independent Brownian motions.

Proposition 2.10. *Every bounded harmonic function on $\mathbb{R}^n$ is constant.*

Proof. For $x, y \in \mathbb{R}^n$, let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two independent Brownian motions starting at x and y respectively. We couple $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ in the following way. They move independently until the first time $\tau_1 := \inf\{t \geq 0 : X_t^1 = Y_t^1\}$. τ_1 is finite $\mathbb{P}$-a.s., since $(X_t^1 - Y_t^1)_{t\geq 0}$ is a one-dimensional Brownian motion starting at $x - y$ and being recurrent, i.e. it achieves 0 eventually $\mathbb{P}$-a.s. (this property will be shown later in Example 2.12). After τ_1, the first coordinates of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ stay together, while the other coordinates continue moving independently until $\tau_2 := \inf\{t > \tau_1 : X_t^2 = Y_t^2\} > \tau_1$. After τ_2, the first and second coordinates stay together. We repeat this process until time $\tau_n := \inf\{t > \tau_{n-1} : X_t^n = Y_t^n\}$. After τ_n, all coordinates stay together, i.e. $X_t = Y_t$ for all $t \geq \tau_n$. If h is a bounded harmonic function on $\mathbb{R}^n$, Theorem 2.9 implies that

$$\begin{aligned}
|h(x) - h(y)| &= |\mathbb{E}h(X_t) - \mathbb{E}h(Y_t)| \\
&\leq \mathbb{E}|h(X_t) - h(Y_t)| \\
&= \mathbb{E}\left[|h(X_t) - h(Y_t)|\mathbb{1}_{\{t<\tau_n\}}\right] + \underbrace{\mathbb{E}\left[|h(X_t) - h(Y_t)|\mathbb{1}_{\{t\geq\tau_n\}}\right]}_{=0} \\
&\leq 2\|h\|_\infty \mathbb{P}(t < \tau_n).
\end{aligned}$$

Letting $t \to \infty$ gives $h(x) = h(y)$, since τ_n is finite $\mathbb{P}$-a.s. ∎

In the following, if not stated otherwise, f is a nonnegative measurable function on ∂D. Furthermore, let τ_S denote the first exit time of Brownian motion from a Borel set S:

$$\tau_S := \inf\{t > 0 : B_t \in S^c\}$$

Recall that τ_S is a stopping time by Proposition 1.5 and 1.7, if S is open or closed.

Here comes the first result involving the (bounded or unbounded) domain D.

Theorem 2.11. *The function*

$$h(x) := \mathbb{E}^x\left[f(B_{\tau_D}), \tau_D < \infty\right]$$

is either harmonic in D or $\equiv \infty$ on D.

Proof. Note that h is well-defined, since $B_{\tau_D} \in \partial D$ by path continuity of Brownian motion and $f(B_{\tau_D})$ is measurable as a composition of two measurable functions.

We suppose h is finite and show the mean value property for h. Hence, take $x \in D$ and $r > 0$ such that $B(x,r) \subset D$. Note that for paths starting at x, and thus inside $B(x,r)$, $\tau_B := \tau_{B(x,r)} < \infty$ by regarding a single coordinate of $(B_t)_{t\geq 0}$ which takes on infinitely large values. Again path continuity of $(B_t)_{t\geq 0}$ implies that paths ω beginning inside $B(x,r)$ satisfy $\tau_D = \tau_B + \tau_D \circ \theta_{\tau_B}$. It follows that

$$\tau_D < \infty \quad \text{if and only if} \quad \tau_D \circ \theta_{\tau_B} < \infty$$

and

$$B_{\tau_D} = B_{\tau_D} \circ \theta_{\tau_B}. \tag{2.8}$$

(2.8) explains why we will lose the shifts by τ_B in the following application of the strong Markov property, Theorem 1.18. For this application, take

$$\tau = \tau_B \quad \text{and} \quad X = f(B_{\tau_D}) \mathbb{1}_{\{\tau_D < \infty\}}.$$

The result is

$$\mathbb{E}^x \left[f(B_{\tau_D}) \mathbb{1}_{\{\tau_D < \infty\}} \,\middle|\, \mathcal{F}_{\tau_B} \right] = \mathbb{E}^{B_{\tau_B}} \left[f(B_{\tau_D}), \tau_D < \infty \right] = h(B_{\tau_B}) \quad \mathbb{P}^x\text{-a.s.}$$

for a bounded function f. Since every nonnegative measurable function can be approximated monotonically by bounded functions, the general nonnegative case follows by applying the monotone convergence theorem for conditional expectations. By taking expectations above, it holds

$$h(x) = \mathbb{E}^x h(B_{\tau_B}). \tag{2.9}$$

Since the distribution of B_{τ_B} is rotationally invariant, it is $\sigma_{x,r}$ by Lemma 2.8. Thus (2.9) becomes the mean value property (2.2).

We assumed h to be finite. Of course, for a nonnegative measurable function f there is a possibility that h takes on infinite values. Let us consider the set

$$H := \{x \in D : h(x) < \infty\}.$$

We will show that H is open and relatively closed implying that h is finite in D or $\equiv \infty$ on D.

Openness: Take $x \in D$ and $r > 0$ such that $B(x,r) \subset D$. By the previous steps, we know that h satisfies the mean value property at x, so we can set $\phi := \mathbb{1}_{[0,r]}$ in (2.3) to get

$$h(x) = \frac{1}{V_n r^n} \int_{B(x,r)} h(z)\, dz \tag{2.10}$$

and choose $y \in D$ with $|x-y| < \frac{r}{2}$ implying $B\left(y, \frac{r}{2}\right) \subset B(x,r)$. We compute

$$\begin{aligned} h(x) &= \frac{1}{V_n r^n} \int_{B(x,r)} h(z)\,dz \geq \frac{1}{V_n r^n} \int_{B\left(y,\frac{r}{2}\right)} h(z)\,dz \\ &= \frac{1}{V_n \left(\frac{r}{2}\right)^n 2^n} \int_{B\left(y,\frac{r}{2}\right)} h(z)\,dz = 2^{-n} h(y), \end{aligned}$$

using the nonnegativity of f and h. Thus, if $h(x) < \infty$, then $h < \infty$ in some open neighborhood of x.
Relative closedness: Take a sequence $(x_n)_{n\in\mathbb{N}}$ in H with $x_n \xrightarrow{n\to\infty} x$. Note that we can assume that $x \in D$, since we want to show closedness in the relative topology of D. Since D is open, there exists $x_{n_0} \in H$, $n_0 \in \mathbb{N}$, with $|x_{n_0} - x| < \frac{\epsilon}{2}$ for some $\epsilon > 0$ such that $B(x_{n_0}, \epsilon) \subset D$. It follows that $B\left(x, \frac{\epsilon}{2}\right) \subset B(x_{n_0}, \epsilon)$. We compute

$$h(x) = \frac{1}{V_n \left(\frac{\epsilon}{2}\right)^n} \int_{B\left(x,\frac{\epsilon}{2}\right)} h(z)\,dz \leq \frac{1}{V_n \epsilon^n} 2^n \int_{B(x_{n_0},\epsilon)} h(z)\,dz = 2^n h(x_{n_0}) < \infty.$$

Hence, H as a subset of a connected set must be either the whole set or the empty set. This says that either h is finite on D or $\equiv \infty$ on D. (In the case that h is finite we proved that h satisfies the mean value property).

If h is finite, we still need to prove continuity of h in D. But this is immediately, since (2.10) gives for $x, y \in D$ and sufficiently small $r > 0$

$$\begin{aligned} |h(x) - h(y)| &= \left| \frac{1}{V_n r^n} \left(\int_{B(x,r)} h(z)\,dz - \int_{B(y,r)} h(z)\,dz \right) \right| \\ &\leq \frac{1}{V_n r^n} \int_{B(x,r) \triangle B(y,r)} h(z)\,dz, \end{aligned} \tag{2.11}$$

where $\triangle$ denotes the symmetric difference of two sets. Thus, $|h(x) - h(y)|$ can be made arbitrarily small by reducing the distance of x and y (making $B(x,r) \triangle B(y,r)$ arbitrarily small). ■

Remark. (a) Until we considered the set H in the proof above, everything holds for bounded f as well. But for bounded f the set H is obviously not empty implying that $H = D$. This means that h is harmonic for bounded f.
(b) Note that we did not assume (un)boundedness of D in the theorem above, which will be very useful in the sequel.

Example 2.12. In the following, we try to apply the knowledge we got from Theorem 2.11 to study the exit behavior of Brownian motion and some consequences.

One-dimensional case: Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and take $0 < r_1 < r_2$. Define the stopping times

$$\tau_{r_1} := \inf\{t > 0 : B_t = r_1\} \quad \text{and} \quad \tau_{r_2} := \inf\{t > 0 : B_t = r_2\}.$$

We want to calculate the probabilities $\mathbb{P}^x(\tau_{r_1} < \tau_{r_2})$ and $\mathbb{P}^x(\tau_{r_2} < \tau_{r_1})$ for a Brownian motion starting at $x \in (r_1, r_2)$. By the optional sampling theorem, $(B_{\tau_{r_1}\wedge\tau_{r_2}\wedge t})_{t\geq 0}$ is a martingale implying

$$x = \mathbb{E}^x B_{\tau_{r_1}\wedge\tau_{r_2}\wedge t}.$$

Since τ_{r_1},τ_{r_2} are $\mathbb{P}^x$-a.s. finite stopping times, letting $t \to \infty$ gives

$$\begin{aligned} x &= \mathbb{E}^x B_{\tau_{r_1}\wedge\tau_{r_2}} \\ &= r_1 \underbrace{\mathbb{P}^x(\tau_{r_1} < \tau_{r_2})}_{=1-\mathbb{P}^x(\tau_{r_2}<\tau_{r_1})} + r_2 \mathbb{P}^x(\tau_{r_2} < \tau_{r_1}) \\ &= r_1 + (r_2 - r_1)\mathbb{P}^x(\tau_{r_2} < \tau_{r_1}). \end{aligned}$$

It follows that

$$\mathbb{P}^x(\tau_{r_2} < \tau_{r_1}) = \frac{x - r_1}{r_2 - r_1} \quad \text{and} \quad \mathbb{P}^x(\tau_{r_1} < \tau_{r_2}) = \frac{r_2 - x}{r_2 - r_1}.$$

Multi-dimensional case: Now, let $(B_t)_{t\geq 0}$ be an n-dimensional Brownian motion. Take again $0 < r_1 < r_2$ and consider the domain

$$D = \{x \in \mathbb{R}^n : r_1 < |x| < r_2\}$$

and boundary values

$$f(x) = \begin{cases} 0 & \text{if } |x| = r_1, \\ 1 & \text{if } |x| = r_2. \end{cases}$$

By Theorem 2.11, the function

$$h_1(x) = \mathbb{E}^x f(B_{\tau_D}) = \mathbb{P}^x(B_t \text{ exits } D \text{ through the outer boundary})$$

is harmonic in D. The results of the next section will show that h_1 satisfies the boundary values in the sense that

$$\lim_{\substack{x\to z \\ x\in D}} h_1(x) = f(z) \qquad , z \in \partial D.$$

Now, let h_2 be the function

$$h_2(x) = \begin{cases} a\log|x| + b & \text{if } n = 2, \\ \frac{a}{|x|^{n-2}} + b & \text{if } n \geq 3. \end{cases}$$

By Example 2.4, h_2 is harmonic in D and also satisfies the boundary values for constants $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ which have to be chosen correctly. On the inner boundary, h_2 has to be zero whereas on the outer boundary it has to be one, i.e. for $n = 2$

$$a \log r_1 + b = 0, \quad a \log r_2 + b = 1$$

and for $n \geq 3$

$$\frac{a}{r_1^{n-2}} + b = 0, \quad \frac{a}{r_2^{n-2}} + b = 1.$$

This leads for $n = 2$ to

$$a = \frac{1}{\log r_2 - \log r_1}, \quad b = \frac{-\log r_1}{\log r_2 - \log r_1}$$

and for $n \geq 3$ to

$$a = \frac{(r_1 r_2)^{n-2}}{r_1^{n-2} - r_2^{n-2}}, \quad b = \frac{r_2^{n-2}}{r_2^{n-2} - r_1^{n-2}}.$$

Since $h_1 = h_2$ on ∂D, we can use Corollary 2.7 to conclude

$$\mathbb{P}^x \left(B_t \text{ exits } D \text{ on the outer boundary} \right) = \begin{cases} \frac{\log|x| - \log r_1}{\log r_2 - \log r_1} & \text{if } n = 2, \\ \left(\frac{r_2}{|x|} \right)^{n-2} \frac{|x|^{n-2} - r_1^{n-2}}{r_2^{n-2} - r_1^{n-2}} & \text{if } n = 3. \end{cases}$$

Recall from the one-dimensional case that this probability for $n = 1$ is given by

$$\frac{x - r_1}{r_2 - r_1} \quad \text{for } r_1 < x < r_2.$$

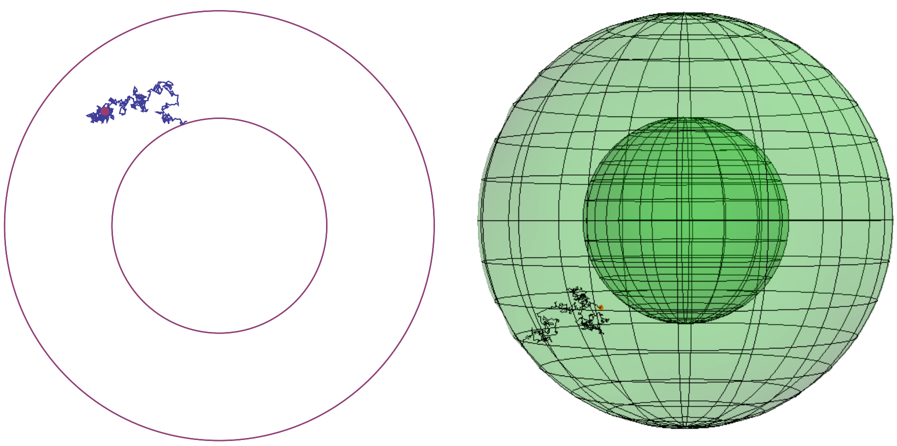

Figure 2.2: Brownian motion stopped at the boundary of D in two and three dimensions

There is an interesting consequence of this example for $n \geq 2$ if we let $r_1 \downarrow 0$ and $r_2 \uparrow \infty$. Define for $r > 0$

$$\tau_r := \inf\{t > 0 : B_t \in \partial B(0,r)\}$$

and note that the probability above can also be written as $\mathbb{P}^x(\tau_{r_2} < \tau_{r_1})$. It follows that for $n \geq 2$

$$\lim_{r_1 \downarrow 0} \mathbb{P}^x(\tau_{r_2} < \tau_{r_1}) = 1 \quad \text{for } 0 < |x| < r_2.$$

Let τ_0 denote the first hitting time of 0 and take a decreasing sequence $(r_1^n)_{n\in\mathbb{N}}$ with $0 < r_1^n < r_2$ for all $n \in \mathbb{N}$ and $r_1^n \xrightarrow{n\to\infty} 0$. Then, we get an increasing sequence of sets $\left(\left\{\tau_{r_2} < \tau_{r_1^n}\right\}\right)_{n\in\mathbb{N}}$ satisfying

$$\{\tau_{r_2} < \tau_0\} = \bigcup_{n\in\mathbb{N}} \left\{\tau_{r_2} < \tau_{r_1^n}\right\}.$$

Using the continuity of the probability measure $\mathbb{P}^x$, this implies

$$\mathbb{P}^x(\tau_{r_2} < \tau_0) = \lim_{r_1 \downarrow 0} \mathbb{P}^x(\tau_{r_2} < \tau_{r_1}) = 1 \quad \text{for } 0 < |x| < r_2.$$

For letting $r_2 \uparrow \infty$, take now an increasing sequence $(r_2^n)_{n\in\mathbb{N}}$ with $r_2^n > 0$ for all $n \in \mathbb{N}$ and $r_2^n \xrightarrow{n\to\infty} \infty$. This time, we get a decreasing sequence of sets $\left(\left\{\tau_{r_2^n} < \tau_0\right\}\right)_{n\in\mathbb{N}}$ satisfying

$$\{\tau_0 = \infty\} \supseteq \bigcap_{n\in\mathbb{N}} \left\{\tau_{r_2^n} < \tau_0\right\}.$$

As above, this implies $\mathbb{P}^x(\tau_0 = \infty) = 1$ for $x \neq 0$, since

$$\mathbb{P}^x(\tau_0 = \infty) \geq \lim_{r_2 \uparrow \infty} \mathbb{P}^x(\tau_{r_2} < \tau_0) = 1.$$

In other words, this means that a Brownian motion starting at a point $x \neq 0$ never hits 0 $\mathbb{P}^x$-a.s.

The case $x = 0$ is still missing. First, let us rewrite the relevant event:

$$\{\tau_0 < \infty\} = \{\exists t > 0 : B_t = 0\} = \bigcup_{n\in\mathbb{N}} \left\{\exists t > \frac{1}{n} : B_t = 0\right\}$$

Now regard the event $\left\{\exists t > \frac{1}{n} : B_t = 0\right\}$ for arbitrary $n \in \mathbb{N}$. The idea is now to wait until time $\frac{1}{n}$, to start a Brownian motion from $B_{1/n}$ (which is $\mathbb{P}^0$-a.s. a positive distance away from 0, since $\mathbb{P}^0\left(B_{1/n}=0\right)=0$) and to use the Markov property then. Thus, it follows

$$\begin{aligned}\mathbb{P}^0\left(\exists t>\frac{1}{n}:B_t=0\right)&=\mathbb{P}^0\left(\exists t>0:B_{t+\frac{1}{n}}=0\right)\\&=\mathbb{E}^0\left[\mathbb{P}^0\left(\exists t>0:B_{t+\frac{1}{n}}=0\,\middle|\,\mathcal{F}_{1/n}\right)\right]\\&=\mathbb{E}^0\left[\mathbb{P}^{B_{1/n}}\left(\exists t>0:B_t=0\right)\right]\\&=\mathbb{E}^0\Big[\underbrace{\mathbb{P}^{B_{1/n}}\left(\tau_0<\infty\right)}_{=0}\Big]=0.\end{aligned}$$

It follows $\mathbb{P}^0\left(\tau_0<\infty\right)=0$. This property can be easily generalized to arbitrary points $x\in\mathbb{R}^n$ (by taking balls with center x instead of 0) and is expressed by saying that Brownian motion does not hit points in dimensions ≥ 2.

Above, we let $r_1\downarrow 0$ first, but we can derive another property of Brownian motion if we only let $r_2\uparrow\infty$. For this, take an increasing sequence $(r_2^n)_{n\in\mathbb{N}}$ with $r_2^n\xrightarrow{n\to\infty}\infty$. Observe that the sequence of sets $(A_n)_{n\in\mathbb{N}}$, defined by $A_n:=\left\{\tau_{r_2^n}<\tau_{r_1}\right\}$, is decreasing and it holds

$$\left\{\tau_{r_1}=\infty,\limsup_{t\to\infty}|B_t|=\infty\right\}=\bigcap_{n\in\mathbb{N}}\left\{\tau_{r_n^2}<\tau_{r_1}\right\}.$$

Take $x\in\mathbb{R}^n$ with $|x|>r_1$. Since $\mathbb{P}^x\left(\limsup_{t\to\infty}|B_t|=\infty\right)=1$, it follows by Lemma A.9,

$$\mathbb{P}^x\left(\tau_{r_1}=\infty\right)=\lim_{r_2\uparrow\infty}\mathbb{P}^x\left(\tau_{r_2}<\tau_{r_1}\right).$$

For $n=2$, we compute

$$\lim_{r_2\uparrow\infty}\mathbb{P}^x\left(\tau_{r_2}<\tau_{r_1}\right)=\lim_{r_2\uparrow\infty}\frac{\log|x|-\log r_1}{\log r_2-\log r_1}=0,$$

and if $n\geq 3$, it holds

$$\begin{aligned}\lim_{r_2\uparrow\infty}\mathbb{P}^x\left(\tau_{r_2}<\tau_{r_1}\right)&=\lim_{r_2\uparrow\infty}\frac{r_2^{n-2}(|x|^{n-2}-r_1^{n-2})}{|x|^{n-2}(r_2^{n-2}-r_1^{n-2})}\\&=\lim_{r_2\uparrow\infty}\frac{(r_2|x|)^{n-2}}{(r_2|x|)^{n-2}-(r_1|x|)^{n-2}}-\lim_{r_2\uparrow\infty}\frac{(r_1r_2)^{n-2}}{(r_2|x|)^{n-2}-(r_1|x|)^{n-2}}\\&=1-\left(\frac{r_1}{|x|}\right)^{n-2}=\frac{|x|^{n-2}-r_1^{n-2}}{|x|^{n-2}}.\end{aligned}$$

In other words, these results say that Brownian motion is neighborhood recurrent (i.e. it hits every ball eventually with probability 1) if $n\geq 2$, but is neighborhood transient (i.e. not neighborhood recurrent) for $n\geq 3$.

Remark. Above we dealt with probabilities of hitting times of circles. Another interesting question is: What is the expected amount of time spent in $B(0,r)$, $r>0$, for a Brownian motion $(B_t)_{t\geq 0}$ started at $x\in B(0,r)$? If we recognize that $(M_t := |B_t|^2 - nt)_{t\geq 0}$ is a martingale, the answer will follow with a few steps. So, choose $x\in B(0,r)$ and let us show that $(M_t)_{t\geq 0}$ is a martingale (with respect to the natural filtration of Brownian motion $(\mathcal{F}_t^0)_{t\geq 0}$):

M_t is integrable for $t\geq 0$ since

$$\mathbb{E}^x|M_t| \leq \mathbb{E}^x|B_t|^2 + nt = \sum_{i=1}^n \mathbb{E}^x(B_t^i)^2 + nt = 2nt < \infty.$$

The adaptedness is obvious. Note that for each $i=1,\ldots,n$, $((B_t^i)^2 - t)_{t\geq 0}$ is a martingale under $\mathbb{P}^x$ by Proposition 1.3 (a). Thus, it follows that for $0\leq s\leq t$

$$\begin{aligned}\mathbb{E}^x\left[M_t \,\middle|\, \mathcal{F}_s^0\right] &= \sum_{i=1}^n \mathbb{E}^x\left[(B_t^i)^2 \,\middle|\, \mathcal{F}_s^0\right] - nt = \sum_{i=1}^n \mathbb{E}^x\left[(B_t^i)^2 - t \,\middle|\, \mathcal{F}_s^0\right] \\ &= \sum_{i=1}^n \left((B_t^i)^2 - s\right) = |B_s|^2 - ns = M_s.\end{aligned}$$

Let now $\tau_B := \tau_{B(0,r)}$ be the first exit time of the ball $B(0,r)$. Since $(M_t)_{t\geq 0}$ is a martingale and τ_B a stopping time, $(M_{\tau_B\wedge t})_{t\geq 0}$ is also a martingale. For $t\geq 0$ this gives

$$\mathbb{E}^x M_{\tau_B\wedge t} = \mathbb{E}^x M_0 = |x|^2.$$

Since $\tau_B<\infty$ $\mathbb{P}^x$-a.s., it follows by the bounded convergence theorem that

$$\mathbb{E}^x M_{\tau_B} = |x|^2.$$

On the other hand, it holds that $\mathbb{E}^x M_{\tau_B} = r^2 - n\,\mathbb{E}^x\tau_B$, so the result is

$$\mathbb{E}^x\tau_B = \frac{r^2 - |x|^2}{n}.$$

2.3 Solving the Dirichlet problem

Definition 2.13. Let $f : \partial D \to \mathbb{R}$ be a function. A function h is called a *solution to the Dirichlet problem on D with boundary values f* if it is harmonic on D and satisfies

$$\lim_{\substack{x \to z \\ x \in D}} h(x) = f(z), \qquad z \in \partial D.$$

So far, we found a harmonic function on D, which is the candidate for the unique solution of the Dirichlet problem on bounded domains. In this case, it remains to show that this candidate is continuous up to the boundary ∂D. And it turns out that proving this property has to take the structure of the boundary and the boundary values f into account.

Recall that $(\mathcal{F}_t^0)_{t \geq 0}$ denotes the natural filtration of Brownian motion whereas $(\mathcal{F}_t)_{t \geq 0}$ stands for its right-continuous extension. Note that the event $\{\tau_D = 0\}$ is in $\mathcal{F}_0$ (but not in $\mathcal{F}_0^0$). Indeed,

$$\{\tau_D = 0\} = \bigcap_{n \in \mathbb{N}} \left\{ \tau_D \leq \frac{1}{n} \right\} \in \bigcap_{n \in \mathbb{N}} \mathcal{F}^0_{1/n} = \mathcal{F}_0.$$

In order to use Blumenthal's 0-1 law, we have to deduce a version for n-dimensional Brownian motion from the one-dimensional statement:

Proposition 2.14. *Blumenthal's* 0-1 *law, as it was stated in Proposition 1.11 (b), also holds in higher dimensions.*

Proof. Recall that

$$\mathcal{F}_0 := \tilde{\mathcal{F}}_0^{\otimes n} = \sigma\left(\left\{ A_1 \times \ldots \times A_n : A_i \in \tilde{\mathcal{F}}_0,\ i \in [n] \right\} \right),$$

where $\tilde{\mathcal{F}}_0$ is the σ-algebra at time 0 of the filtration used in the one-dimensional case. Since

$$\mathcal{P} := \left\{ A_1 \times \ldots \times A_n : A_i \in \tilde{\mathcal{F}}_0,\ i \in [n] \right\}$$

is a π-system that generates $\mathcal{F}_0$, it suffices to show that

(a) $\mathcal{L}_x := \{A \in \mathcal{F}_0 : \mathbb{P}^x(A) = 0 \text{ or } 1\}$ is a λ-system for every $x \in \mathbb{R}^n$,

(b) $\mathcal{P} \subset \mathcal{L}_x$ for every $x \in \mathbb{R}^n$

by using the π-λ-theorem, Theorem A.7, for $\mathcal{P}$ and $\mathcal{L}$.

First, we prove (a). Take arbitrary $x \in \mathbb{R}^n$ and show the defining three properties of a λ-system (Definition A.6):

1. $\Omega \in \mathcal{L}_x$, since $\mathbb{P}^x(\Omega) = 1$.
2. Take $A, B \in \mathcal{L}_x$ with $A \subset B$. It holds
$$\mathbb{P}^x(B \backslash A) = \mathbb{P}^x(B) - \mathbb{P}^x(A) = 0 \text{ or } 1.$$
So, $B \backslash A \in \mathcal{L}_x$.
3. Take $(A_n)_{n \in \mathbb{N}} \subset \mathcal{L}_x$ with $A_n \uparrow A$. It holds
$$\mathbb{P}^x(A) = \lim_{n \to \infty} \mathbb{P}^x(A_n) = 0 \text{ or } 1.$$
So, $A \in \mathcal{L}_x$.

For (b), take $A \in \mathcal{P}$. Then, there exist sets $A_1, \ldots, A_n \in \tilde{\mathcal{F}}_0$ such that
$$A = A_1 \times \ldots \times A_n.$$
Since $A_1, \ldots, A_n$ are independent by definition (see beginning of Section 2.2), we get
$$\mathbb{P}^x(A) = \tilde{\mathbb{P}}^{x_1}(A_1) \cdot \ldots \cdot \tilde{\mathbb{P}}^{x_n}(A_n) = 0 \text{ or } 1,$$
where $\tilde{\mathbb{P}}^{x_i}$, $i \in [n]$, are the corresponding probability measures from the one-dimensional case. Thus, $A \in \mathcal{L}_x$. ■

So, by Blumenthal's 0-1 law, it follows for every $x \in \mathbb{R}^n$ that $\mathbb{P}^x(\tau_D = 0) = 0$ or 1. Obviously, for $x \in D$ it holds $\mathbb{P}^x(\tau_D) = 0$ and $\mathbb{P}^x(\tau_D) = 1$ for $x \notin \overline{D}$ (both by path continuity of Brownian motion). Thus, the interesting case is $x \in \partial D$. In general, it depends on the boundary point whether $\mathbb{P}^x(\tau_D = 0)$ is 0 or 1. This fact justifies the following definition.

Definition 2.15. A point $x \in \partial D$ is called *regular* if $\mathbb{P}^x(\tau_D = 0) = 1$ and *irregular* if $\mathbb{P}^x(\tau_D = 0) = 0$.

Example 2.16. (a) Let $D = \{x \in \mathbb{R}^n : x_n > 0\}$. Since B_t^n is a one-dimensional Brownian motion, Corollary 1.12 implies that all points $x \in \partial D$ are regular.

(b) For $n \geq 2$ and $D = \{x \in \mathbb{R}^n : 0 < |x| < 1\}$, 0 is irregular, since the probability that Brownian motion immediately returns to 0 is 0 (Brownian motion does not hit points for $n \geq 2$). For $n = 1$, 0 is regular by Corollary 1.12 again.

(c) Consider $D = \{x \in \mathbb{R}^n : |x| < 1\} \setminus \{x \in \mathbb{R}^n : x_1 \geq 0, x_2 = \ldots = x_n = 0\}$. Recall that Brownian motion does not hit points in dimensions ≥ 2. It follows that 0 is irregular for $n \geq 3$ since $(B_t^2, \ldots, B_t^n)_{t\geq 0}$ as an $(n-1)$-dimensional Brownian motion does not hit 0. In the case $n = 2$, 0 is regular. Indeed, let $\sigma_n := \inf\left\{t > \frac{1}{n} : B_t^2 = 0\right\}$ and $\tau_0 := \inf\{t > 0 : B_t^2 = 0\}$. Note that $\lim_{n\to\infty} \sigma_n = \tau_0 = 0$ $\mathbb{P}^0$-a.s. by Corollary 1.12. Since $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are two independent Brownian motions, $B_{\sigma_n}^1$ is a symmetric random variable implying

$$\mathbb{P}^0\left(B_{\sigma_n}^1 \geq 0\right) = \mathbb{P}^0\left(B_{\sigma_n}^1 \leq 0\right) \geq \frac{1}{2}.$$

Recognizing that $\{\tau_D \leq \sigma_n\} \supseteq \left\{B_{\sigma_n}^1 \geq 0\right\}$, we get

$$\mathbb{P}^0(\tau_D = 0) = \lim_{n\to\infty} \mathbb{P}^0(\tau_D \leq \sigma_n) \geq \liminf_{n\to\infty} \mathbb{P}^0\left(B_{\sigma_n}^1 \geq 0\right) \geq \frac{1}{2}.$$

Since $\{\tau_D = 0\} \in \mathcal{F}_0$, we can apply Blumenthal's 0-1 law, which gives

$$\mathbb{P}^0(\tau_D = 0) = 1.$$

Since the notion of regularity is not very handy at first sight, we introduce another condition being sufficient for regularity, the *truncated cone condition*. A *cone* is a set of points obtained by rotating the set

$$\left\{z \in \mathbb{R}^n : 0 < z_1,\ z_2^2 + \cdots + z_n^2 \leq cz_1^2\right\},$$

where $c > 0$ can be seen as the "width" of the cone. To get a truncated cone we just have to add the condition $z_1 < r$ for $r > 0$ denoting the "length" of the cone. A cone with vertex at z can be obtained by translating a cone with vertex at the origin by z.

Definition 2.17. A point $z \in \partial D$ satisfies the *truncated cone condition* if there exists a truncated cone C_0 with vertex at z such that $C_0 \subseteq D^c$.

Example 2.18. (a) Let $D = \{x \in \mathbb{R}^n : |x| < 1\}$. Obviously, every $z \in \partial D$ satisfies the truncated cone condition.

(b) Take $D = B(-1,1) \cup B(1,1)$. Then $z = 0$ does not satisfy the truncated cone condition.

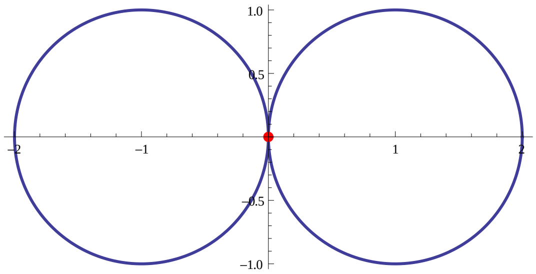

Figure 2.3: Example 2.18 (b) in two dimensions

Lemma 2.19. *If $z \in \partial D$ satisfies the truncated cone condition, then z is regular.*

Proof. Let $z \in \partial D$ satisfy the truncated cone condition by a truncated cone C_0 and let C denote the corresponding full cone. We have to show that

$$\mathbb{P}^z\left(\tau_D = 0\right) = 1. \tag{2.12}$$

Since $\{\tau_D = 0\} \in \mathcal{F}_0$, it suffices to prove that $\mathbb{P}^z\left(\tau_D = 0\right) > 0$, since (2.12) will follow by Blumenthal's 0-1 law. First, regard $\mathbb{P}^z\left(\tau_D \leq t\right)$ for some $t > 0$. We compute

$$\mathbb{P}^z\left(\tau_D \leq t\right) \geq \mathbb{P}^z\left(B_t \in C_0\right) = \mathbb{P}^z\left(B_t \in C\right) - \mathbb{P}^z\left(B_t \in C \backslash C_0\right). \tag{2.13}$$

Note that $\mathbb{P}^z\left(B_t \in C\right)$ is independent of t. To see this, take random variables Z_i, $i \in [n]$, which are all $\mathcal{N}(z,1)$-distributed under $\mathbb{P}^z$. Then for $i \in [n]$, $\sqrt{t}Z_i$ is $\mathcal{N}(z,t)$-distributed under $\mathbb{P}^z$ as B_t^i is, as well. It follows

$$\begin{aligned}
\mathbb{P}^z\left(B_t \in C\right) &\overset{(*)}{=} \mathbb{P}^z\left(\frac{(B_t^2)^2 + \cdots + (B_t^n)^2}{(B_t^1)^2} \leq c\right) \\
&= \mathbb{P}^z\left(\frac{(\sqrt{t}Z_2)^2 + \cdots + (\sqrt{t}Z_n)^2}{(\sqrt{t}Z_1)^2} \leq c\right) \\
&= \mathbb{P}^z\left(\frac{Z_2^2 + \cdots + Z_n^2}{Z_1^2} \leq c\right),
\end{aligned}$$

which is independent of t. At $(*)$, we used the rotational invariance of Brownian motion. Furthermore, note that this probability is strictly positive, since C has strictly positive Lebesgue-measure and B_t is multi-dimensional normally distributed. Since

$$\mathbb{P}^z\left(B_t \in C \backslash C_0\right) \xrightarrow{t \downarrow 0} 0,$$

we can conclude that $\mathbb{P}^z\left(\tau_D = 0\right) = \lim_{t \downarrow 0} \mathbb{P}^z\left(\tau_D \leq t\right) > 0$ by letting $t \downarrow 0$ in (2.13). ■

Lemma 2.20. *For $t > 0$, the function $x \mapsto \mathbb{P}^x(\tau_D \le t)$ is lower semicontinuous on $\mathbb{R}^n$, i.e.*

$$\liminf_{x \to z} \mathbb{P}^x(\tau_D \le t) \ge \mathbb{P}^z(\tau_D \le t), \quad z \in \mathbb{R}^n.$$

Proof. We try to construct an increasing sequence of continuous functions that converges pointwise to $x \mapsto \mathbb{P}^x(\tau_D \le t)$, because then we can use the fact that an increasing limit of continuous functions is lower semicontinuous (see Lemma A.5).

Fix $0 < s < t$. By the Markov property, we can write

$$\begin{aligned}
\mathbb{P}^x(\exists u \in (s,t] : B_u \in D^c) &= \mathbb{E}^x\left[\mathbb{P}^x(\exists u \in (0,t-s] : B_{u+s} \in D^c \,|\, \mathcal{F}_s)\right] \\
&= \mathbb{E}^x\left[\mathbb{P}^{B_s}(\exists u \in (0,t-s] : B_u \in D^c)\right] \\
&= \mathbb{E}^x\left[\mathbb{P}^{B_s}(\tau_D \le t-s)\right] \\
&= \int_{\mathbb{R}^n} p_s(x,y)\, \mathbb{P}^y(\tau_D \le t-s)\, dy,
\end{aligned} \tag{2.14}$$

where $p_s(x,y) := \frac{1}{\sqrt{2\pi s}} e^{-\frac{|x-y|^2}{2s}}$ denotes the density of B_s, if $B_0 = x$. The right side of (2.14) is continuous in x, thus the left side is, as well.

It remains to show that $\mathbb{P}^x(\tau_D \le t)$ is the increasing limit of the left side of (2.14) as $s \downarrow 0$. For this, take a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \downarrow 0$ as $n \to \infty$. Note that the sequence of sets $(A_n)_{n\in\mathbb{N}}$, defined by $A_n := \{\exists u \in (s_n, t] : B_u \in D^c\}$, is increasing and satisfies

$$\{\tau_D \le t\} = \{\exists u \in (0,t] : B_u \in D^c\} = \bigcup_{n\in\mathbb{N}} A_n.$$

By the continuity of the measure $\mathbb{P}^x$, it follows that

$$\mathbb{P}^x(A_n) \uparrow \mathbb{P}^x(\tau_D \le t) \quad \text{as } n \to \infty.$$

■

Now we can state and prove the main result on attainment of boundary values.

Theorem 2.21. *Let $f : \partial D \to \mathbb{R}$ be bounded. If $z \in \partial D$ is regular and f is continuous at z, then*

$$\lim_{\substack{x\to z\\ x\in D}} h(x) = \lim_{\substack{x\to z\\ x\in D}} \mathbb{E}^x\left[f(B_{\tau_D}), \tau_D < \infty\right] = f(z).$$

Proof. Our goal is to show that

$$\limsup_{\substack{x\to z\\ x\in D}} |h(x) - f(z)| = 0.$$

For $r > 0$, let τ_r be the hitting time of $B(z,r)$ and $x \in D \cap B(z,r)$. It follows

$$\begin{aligned} &\mathbb{E}^x\left[|f(B_{\tau_D}) - f(z)|, \tau_D < \infty\right] \\ &\quad = \mathbb{E}^x\left[|f(B_{\tau_D}) - f(z)|, \tau_r \leq \tau_D < \infty\right] + \mathbb{E}^x\left[|f(B_{\tau_D}) - f(z)|, \tau_D \leq \tau_r\right] \\ &\quad \leq 2\|f\|_\infty \mathbb{P}^x\left(\tau_r \leq \tau_D < \infty\right) + \sup_{\substack{|y-z|\leq r \\ y \in \partial D}} |f(y) - f(z)|\, \mathbb{P}^x\left(\tau_D \leq \tau_r\right). \end{aligned} \tag{2.15}$$

Since $\tau_r > 0$ $\mathbb{P}^x$-a.s., there exists a $t > 0$ such that $\mathbb{P}^x(\tau_r \geq t) > 0$. Indeed, for a contradiction, assume that $\mathbb{P}^x(\tau_r \geq t) = 0$ for every $t > 0$. Since

$$\{\tau_r > 0\} = \bigcup_{t \in \mathbb{Q}^+} \{\tau_r \geq t\},$$

it holds that

$$\mathbb{P}^x(\tau_r > 0) = \lim_{t \downarrow 0} \underbrace{\mathbb{P}^x(\tau_r \geq t)}_{=0} = 0,$$

which yields the contradiction to $\tau_r > 0$ $\mathbb{P}^x$-a.s.

Now, we want to show that

$$\limsup_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_D \leq \tau_r) = 1 \tag{2.16}$$

implying

$$\begin{aligned} \limsup_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_r \leq \tau_D < \infty) &\leq \limsup_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_r \leq \tau_D) \\ &= 1 - \limsup_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_D \leq \tau_r) = 0. \end{aligned}$$

For the specific $t > 0$ from above, we compute

$$\begin{aligned} \mathbb{P}^x(\tau_D \leq \tau_r) &= \mathbb{P}^x(\underbrace{\tau_D \leq \tau_r, \tau_r \geq t}_{\supseteq \{\tau_D \leq t\}}) + \underbrace{\mathbb{P}^x(\tau_D \leq \tau_r, \tau_r < t)}_{\geq 0} \\ &\geq \mathbb{P}^x(\tau_D \leq t). \end{aligned}$$

By Lemma 2.20 and the regularity of z, it follows

$$\begin{aligned} \limsup_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_D \leq \tau_r) &\geq \liminf_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_D \leq \tau_r) \\ &\geq \liminf_{\substack{x \to z \\ x \in D}} \mathbb{P}^x(\tau_D \leq t) \\ &\geq \mathbb{P}^z(\tau_D \leq t) = 1, \end{aligned}$$

which gives (2.16).

Note that, again Lemma 2.20 and the regularity of z give

$$\begin{aligned}\limsup_{x\to z} \mathbb{P}^x(\tau_D<\infty) &\ge \limsup_{x\to z} \mathbb{P}^x(\tau_D\le s)\\ &\ge \liminf_{x\to z} \mathbb{P}^x(\tau_D\le s) \ge \mathbb{P}^z(\tau_D\le s)=1\end{aligned} \tag{2.17}$$

for some $s>0$.

Finally, (2.16), (2.17) and passing to the limit in (2.15) give

$$\begin{aligned}\limsup_{\substack{x\to z\\ x\in D}} |h(x)-f(z)| &= \limsup_{\substack{x\to z\\ x\in D}} |h(x)-f(z)\,\mathbb{P}^x(\tau_D<\infty)|\\ &\le \limsup_{\substack{x\to z\\ x\in D}} \mathbb{E}^x\left[|f(B_{\tau_D})-f(z)|, \tau_D<\infty\right]\\ &\le \sup_{\substack{|y-z|\le r\\ y\in\partial D}} |f(y)-f(z)|.\end{aligned}$$

Letting $r\downarrow 0$ and using the continuity of f at z completes the proof. ■

Let us now look at the simplest result about Dirichlet problems for bounded D:

Theorem 2.22. *Suppose D is bounded, every point on ∂D is regular and $f:\partial D\to\mathbb{R}$ is continuous. Then the unique solution to the Dirichlet problem is given by*

$$h(x) := \mathbb{E}^x f(B_{\tau_D}), \quad x\in\overline{D}.$$

Proof. Since D is bounded, it holds $\tau_D<\infty$ $\mathbb{P}^x$-a.s. for every $x\in\overline{D}$. Since D is bounded (and hence ∂D is compact) and f is continuous, f is bounded. Theorem 2.11 and Theorem 2.21 give that h is a solution to the Dirichlet problem. Since f is continuous, h is continuous on $\overline{D}$. Thus, the uniqueness follows from Corollary 2.7. ■

Remark. There is an interesting way to prove the following statement: *If h is a bounded solution to the Dirichlet problem on bounded D with boundary values f, then*

$$h(x)=\mathbb{E}^x f(B_{\tau_D}), \qquad x\in D. \tag{2.18}$$

We see this by Proposition 1.14 and the subsequent remark. Indeed, let $x\in D$. Since

$$h(B_{\tau_D\wedge t}) - \int_0^{\tau_D\wedge t}\frac{1}{2}\Delta h(B_s)\,ds, \qquad t\ge 0$$

is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ under $\mathbb{P}^x$, we can write

$$\mathbb{E}^x h(B_{\tau_D\wedge t}) - \mathbb{E}^x \int_0^{\tau_D\wedge t} \frac{1}{2} \underbrace{\Delta h(B_s)}_{=0} \, ds = \mathbb{E}^x h(B_0) = h(x)$$

$$\iff \qquad h(x) = \mathbb{E}^x h(B_{\tau_D\wedge t}).$$

Since h is bounded and continuous, and $\tau_D < \infty$ $\mathbb{P}^x$-a.s., letting $t\uparrow\infty$ gives

$$h(x) = \mathbb{E}^x h(B_{\tau_D}).$$

Note that $B_{\tau_D} \in \partial D$ by path continuity of Brownian motion, and hence

$$h(B_{\tau_D}) = f(B_{\tau_D}),$$

which gives (2.18).

Example 2.23. Let $n \geq 2$ and $D := \{x \in \mathbb{R}^n : 0 < |x| < 1\}$. Example 2.16(b) gives that the boundary point 0 is not regular. To find a solution to the Dirichlet problem on D with continuous boundary values f, we first want to find a solution to the Dirichlet problem on $\tilde{D} := D \cup \{0\}$ with continuous boundary values $\tilde{f} := f|_{\partial\tilde{D}}$. By Theorem 2.22, the unique solution on $\tilde{D}$ is

$$\tilde{h}(x) = \mathbb{E}^x \tilde{f}(B_{\tau_{\tilde{D}}}), \qquad x \in \tilde{D}.$$

Hence, a good candidate for the solution on D is

$$h(x) := \mathbb{E}^x f(B_{\tau_D}), \qquad x \in D.$$

But this function can only be continuous at 0 if

$$f(0) = \tilde{h}(0) = \mathbb{E}^0 \tilde{f}(B_{\tau_{\tilde{D}}}) = \int_{\partial B(0,1)} \tilde{f}(y)\, \sigma_{0,1}(dy) = \int_{\partial B(0,1)} f(y)\, \sigma_{0,1}(dy).$$

If this is not the case, there is no solution to the Dirichlet problem on D. If it is, the solution is unique by Corollary 2.7.

Next, we want to find out how to deal with unbounded domains. The function

$$g(x) := \mathbb{P}^x\left(\tau_D = \infty\right), \qquad x \in D \tag{2.19}$$

plays an important role for this.

Proposition 2.24. *For g, defined by (2.19), it holds*

(a) g is harmonic in D.

(b) If $z \in \partial D$ is regular, then

$$\lim_{\substack{x \to z \\ x \in D}} g(x) = 0.$$

Proof. Let $f \equiv 1$ on ∂D. First, we prove (a). It holds

$$g(x) = 1 - \mathbb{P}^x(\tau_D < \infty) = 1 - \mathbb{E}^x\left[f(B_{\tau_D}), \tau_D < \infty\right].$$

By Theorem 2.11, g is harmonic in D.

Now prove (b). Let $z \in \partial D$ be regular. We compute

$$\begin{aligned}\lim_{\substack{x \to z \\ x \in D}} g(x) &= 1 - \lim_{\substack{x \to z \\ x \in D}} \mathbb{E}^x\left[f(B_{\tau_D}), \tau_D < \infty\right] \\ &= 1 - f(z) = 0\end{aligned}$$

using Theorem 2.21. ∎

As a consequence, if all boundary points are regular, then g is a solution to the Dirichlet problem with boundary values 0. Using Corollary 2.7, the question of uniqueness reduces to the question whether $g \equiv 0$ on D or not. The answer to this question needs the notion of a *recurrent* set.

Definition 2.25. A Borel set $A \subset \mathbb{R}^n$ is called *recurrent*, if it holds

$$\mathbb{P}^x(\forall t > 0\ \exists s > t : B_s \in A) = 1$$

for every $x \in \mathbb{R}^n$. If A is not recurrent, it is called *transient*.

Example 2.26. Let $D := \{x \in \mathbb{R}^n : |x| > 1\}$. For $n = 2$, D^c is recurrent. On the other hand, if $n \geq 3$, D^c is transient (see Example 2.12).

Theorem 2.27. *Suppose every boundary point is regular. Then $g \equiv 0$ in D if and only if D^c is recurrent.*

Proof. Assume D^c is recurrent. Then $\mathbb{P}^x(\tau_D < \infty) = 1$ for every $x \in \mathbb{R}^n$, so $g \equiv 0$ in D.

Now assume $g \equiv 0$ in D implying $\mathbb{P}^x(\tau_D < \infty) = 1$ for $x \in D$ — in fact for $x \in \mathbb{R}^n$, on ∂D by the regularity assumption and on $\overline{D}^c$ by path continuity of

Brownian motion. To prove recurrence of D^c, fix $x \in \mathbb{R}^n$ and observe that for D as a domain (i.e. open and connected), it holds that

$$\begin{aligned}\mathbb{P}^x\left(\forall t > 0\, \exists s > t : B_s \in D^c\right) &= \mathbb{P}^x\left(\forall t \in \mathbb{Q}^+\, \exists s > t : B_s \in D^c\right)\\ &= \mathbb{P}^x\Big(\bigcap_{t\in\mathbb{Q}^+} \{\exists s > t : B_s \in D^c\}\Big)\end{aligned}$$

again by path continuity. This implies that we need to show

$$\mathbb{P}^x\left(\exists s > t : B_s \in D^c\right) = 1 \tag{2.20}$$

only for a single fixed $t \in \mathbb{Q}^+$. So, fix $t \in \mathbb{Q}^+$. The Markov property gives

$$\mathbb{P}^x\left(\exists s > t : B_s \in D^c \,|\, \mathcal{F}_t\right) = \mathbb{P}^{B_t}\underbrace{\left(\exists s > 0 : B_s \in D^c\right)}_{=\{\tau_D<\infty\}} = 1,$$

where we used in the last step that $\mathbb{P}^x\left(\tau_D < \infty\right) = 1$ for every $x \in \mathbb{R}^n$, which was proved above. Taking expectations on both sides yields (2.20). ■

The next theorem is the main result for unbounded domains.

Theorem 2.28. *Suppose all points on ∂D are regular and f is a bounded and continuous function on ∂D. Then every bounded solution to the Dirichlet problem on D with boundary values f has the form*

$$h(x) := \mathbb{E}^x\left[f(B_{\tau_D}), \tau_D < \infty\right] + c\,\mathbb{P}^x\left(\tau_D = \infty\right) \tag{2.21}$$

for some constant c.

Remark. (a) Note that c is independent of x.
(b) We can regard c as the boundary value of h at ∞.
(c) The boundedness assumption on the solution h in the theorem above is necessary. If $D = \mathbb{R}^n$, nonnegativity is enough by Proposition 2.5, but if $D \neq \mathbb{R}^n$, it is not by the following example: Let

$$D = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

be the upper half space in $\mathbb{R}^n$ and $f \equiv 0$ on ∂D. Then $h(x) = x_n$ is an unbounded, nonnegative solution to the Dirichlet problem, but is not of the form (2.21).

Proof of Theorem 2.28. First note that every function of the form (2.21) is a solution to the Dirichlet problem by Theorem 2.11, Theorem 2.21 and Proposition 2.24.

Now, assume that h is a bounded solution to the Dirichlet problem. If $D = \mathbb{R}^n$, then $\mathbb{P}^x(\tau_D < \infty) = 0$ and $\mathbb{P}^x(\tau_D = \infty) = 1$ for every $x \in \mathbb{R}^n$. So, (2.21) follows by Proposition 2.10. Hence, we assume that $D \neq \mathbb{R}^n$. Let us define $h = f$ on ∂D, such that h becomes continuous on $\overline{D}$. We now have to find a constant c such that (2.21) holds.

We will construct an increasing sequence of bounded, "good" domains D_n that approximate D from the inside and then use some previous results. To describe the sequence, we will need the notion for the distance of a point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ once more, which was originally defined by

$$\operatorname{dist}(x, A) := \inf \{|x - y| : y \in A\}.$$

Note that $\operatorname{dist}(x, A)$ is continuous in x. Now, for $n \in \mathbb{N}$, define

$$D_n := \left\{ x \in D : |x| < n \text{ and } \operatorname{dist}(x, D^c) > \frac{1}{n} \right\},$$

which has the following properties:

(a) D_n is open (but not necessarily connected):
Take $x \in D_n$ and define $\delta := \min \left\{ n - |x|, \operatorname{dist}(x, D^c) - \frac{1}{n} \right\} > 0$. Now, note that for every $y \in B(x, \frac{\delta}{2})$, it holds $y \in D_n$.

(b) D_n is bounded:
Note that $D_n \subset B(0, n)$.

(c) All points on ∂D_n satisfy the cone condition, and therefore are regular by Lemma 2.19:
Take $x \in \partial D_n$. It holds $|x| = n$ or $\operatorname{dist}(x, D^c) = \frac{1}{n}$. Since D_n^c contains all y with $y \geq n$, the cone condition certainly holds in the first case. In the second case, note that there exists a $y \in D^c$ such that $|x - y| = \frac{1}{n}$, since D^c is closed. Take a $z \in B(y, \frac{1}{n})$ (i.e. $|z - y| \leq \frac{1}{n}$) implying $\operatorname{dist}(z, D^c) \leq \frac{1}{n}$ which implies $z \in D_n^c$. It follows that the open ball $\left\{ z \in \mathbb{R}^n : |z - y| < \frac{1}{n} \right\}$ touches x and is fully contained in D_n^c, so the cone condition is satisfied.

Now, it holds that

$$h(x) = \mathbb{E}^x h(B_{\tau_{D_n}}), \quad x \in D_n. \tag{2.22}$$

We can see this by noting that the left side is harmonic on D_n by assumption and the right side is harmonic by Theorem 2.11. Both sides have boundary values h —

the right side by Theorem 2.21. Since D_n is bounded, we can apply Corollary 2.7 to every connected component of it to get (2.22). Note that

$$\mathbb{E}^x h(B_{\tau_{D_{n+1}}}) = h(x) = \mathbb{E}^x h(B_{\tau_{D_n}}), \quad x \in \overline{D_n}, \tag{2.23}$$

since $D_n \subset D_{n+1}$. The strong Markov property and (2.23) give

$$\begin{aligned}\mathbb{E}^x \left[h(B_{\tau_{D_{n+1}}}) \,\middle|\, \mathcal{F}_{\tau_{D_n}} \right] &= \mathbb{E}^{B_{\tau_{D_n}}} h(B_{\tau_{D_{n+1}}}) \\ &= \mathbb{E}^{B_{\tau_{D_n}}} h(B_{\tau_{D_n}}) = h(B_{\tau_{D_n}}), \quad x \in D_n.\end{aligned}$$

This means that, for $k \in \mathbb{N}$, $M_k := (h(B_{\tau_{D_n}}))_{n \geq k}$ is a bounded martingale with respect to $\mathbb{P}^x$, if $x \in D_k$. For $x \in D$, choose $k \in \mathbb{N}$ large enough such that $x \in D_k$ and use the convergence theorem for discrete time martingales, Theorem A.12, to see that

$$Z = \lim_{n \to \infty} h(B_{\tau_{D_n}})$$

exists $\mathbb{P}^x$-a.s. and is in L^1 with respect to $\mathbb{P}^x$. Hence, passing to the limit in (2.22) leads to

$$h(x) = \mathbb{E}^x Z = \mathbb{E}^x \left[Z, \tau_D < \infty \right] + \mathbb{E}^x \left[Z, \tau_D = \infty \right], \quad x \in D.$$

Let us find out how Z behaves on $\{\tau_D < \infty\}$. For this, first note that $\tau_{D_n} \downarrow \tau_D$. Indeed, define $\sigma := \lim_{n \to \infty} \tau_{D_n}$. Since $\tau_{D_n} \leq \tau_D$ for every $n \in \mathbb{N}$, it holds that $\sigma \leq \tau_D$. Now, show that $B_\sigma \in D^c$ implying $\tau_D \leq \sigma$ which yields $\sigma = \tau_D$. It holds, for $k \leq m$,

$$B_{\tau_{D_m}} \in D_m^c \subseteq D_k^c.$$

It follows by path continuity,

$$B_\sigma = \lim_{n \to \infty} B_{\tau_{D_n}} \in \bigcap_{n \in \mathbb{N}} D_n^c \overset{(*)}{=} D^c,$$

where we used the closedness of D^c at $(*)$ for the equality. It follows that

$$h(B_{\tau_{D_n}}) \xrightarrow{n \to \infty} h(B_{\tau_D}) = f(B_{\tau_D}) \ \ \mathbb{P}^x\text{-a.s. on } \{\tau_D < \infty\}$$

for every $x \in D$ by the continuity of h on $\overline{D}$. This gives

$$h(x) = \mathbb{E}^x \left[f(B_{\tau_D}), \tau_D < \infty \right] + \mathbb{E}^x \left[Z, \tau_D = \infty \right], \quad x \in D.$$

To make the proof complete, we have to show that there exists a constant c, independent of x, such that

$$Z = c \ \ \mathbb{P}^x\text{-a.s. on } \{\tau_D = \infty\}$$

for every $x \in D$. For this, we may assume that D^c is transient, since $\mathbb{P}^x\left(\tau_D = \infty\right) = 0$ for all $x \in D$ if D^c is recurrent and the result follows immediately in this case. First of all, we need to prove that

$$L := \lim_{t\to\infty} h(B_t) \text{ exists } \mathbb{P}^x\text{-a.s. for every } x \in D \text{ on } \{\tau_D = \infty\}. \tag{2.24}$$

In doing so, it suffices to show that $(h(B_{\tau_D \wedge t}))_{t\geq 0}$ is a bounded martingale, because (2.24) follows by the convergence theorem for continuous time martingales, Theorem A.13. The boundedness is clear, since h was assumed to be bounded. To check the martingale property, let $h_n \in C^2$ be a function with compact support $\subset D$ such that $h_n = h$ on D_n. Note that h_n is harmonic in D_n, since h was assumed to be harmonic in D. Take $x \in D_n$ and use Proposition 1.14 to see that

$$h_n(B_t) - \int_0^t \frac{1}{2}\Delta h_n(B_s)\, ds, \qquad t \geq 0$$

is a martingale with respect to $\mathbb{P}^x$. By the stopping time theorem,

$$h_n(B_{\tau_{D_n}\wedge t}) - \int_0^{\tau_{D_n}\wedge t} \frac{1}{2} \underbrace{\Delta h_n(B_s)}_{=0}\, ds = h_n(B_{\tau_{D_n}\wedge t}), \qquad t \geq 0$$

is also a martingale with respect to $\mathbb{P}^x$. Since $h_n = h$ on D_n, it follows that $(h(B_{\tau_{D_n}\wedge t}))_{t\geq 0}$ is a martingale with respect to $\mathbb{P}^x$. So, letting $n \to \infty$ yields that $(h(B_{\tau_D\wedge t}))_{t\geq 0}$ is a martingale with respect to $\mathbb{P}^x$ for every $x \in D$.

For completeness, (2.24) holds without restriction to $\{\tau_D = \infty\}$. To see this, first observe that

$$B_t \in D \text{ eventually } \mathbb{P}^x\text{-a.s. for every } x \in D. \tag{2.25}$$

This is by regarding the set

$$A := \left\{\exists (t_n)_{n\in\mathbb{N}} \text{ with } t_n \uparrow \infty : B_{t_n} \in D^c\right\}.$$

Recall that the tail-σ-algebra was defined by $\mathcal{T} := \bigcap_{t>0} \mathcal{F}_t^*$, where $\mathcal{F}_t^*$ is the smallest σ-algebra with respect to which the projection $\omega \mapsto \omega(s) = B_s(\omega)$ is measurable for each $s \geq t$. Note that $A \in \mathcal{T}$ is a tail event and by Proposition 1.13, this means that $\mathbb{P}^x\left(A\right) = 0$ for all $x \in \mathbb{R}^n$ or $\mathbb{P}^x\left(A\right) = 1$ for all $x \in \mathbb{R}^n$. Since D^c was assumed to be transient, $\mathbb{P}^x\left(A\right) < 1$ implying $\mathbb{P}^x\left(A\right) = 0$ for all $x \in \mathbb{R}^n$, (2.25) follows.

Now, fix $x \in D$ and compute

$$\begin{aligned}
\mathbb{P}^x\left(\forall s > t : B_s \in D\right) &= \mathbb{E}^x\left[\mathbb{P}^x\left(\forall s > t : B_s \in D\right) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^x\left[\mathbb{P}^{B_t}\left(\forall s > 0 : B_s \in D\right)\right] \\
&= \mathbb{E}^x\left[\mathbb{P}^{B_t}\left(\tau_D = \infty\right)\right] \\
&\leq \mathbb{E}^x\left[\mathbb{P}^{B_t}\left(L \text{ exists}\right)\right] \\
&= \mathbb{E}^x\left[\mathbb{P}^x\left(L \text{ exists} \mid \mathcal{F}_t\right)\right] \\
&= \mathbb{P}^x\left(L \text{ exists}\right),
\end{aligned} \tag{2.26}$$

where we applied the Markov property two times. The left side of (2.26) tends to 1 as $t \to \infty$ by (2.25) implying that L exists $\mathbb{P}^x$-a.s.

By noting that L is a tail variable (i.e. L is $\mathcal{T}$-measurable), Proposition 1.13 implies that there exists a constant c, independent of x, such that $\mathbb{P}^x\left(L = c\right) = 1$. Therefore, $Z = L = c$ on $\{\tau_D = \infty\}$, which completes the proof. ■

2.4 The Poisson equation

Slightly different to the Dirichlet problem is the problem involving the Poisson equation:

$$\frac{1}{2}\Delta h = -f \quad \text{and} \quad \lim_{\substack{x \to z \\ x \in D}} h(x) = 0 \;\text{ for } z \in \partial D. \tag{2.27}$$

Solutions to (2.27) can be represented by the probabilistic form

$$h(x) = \mathbb{E}^x \int_0^{\tau_D} f(B_s)\, ds, \qquad x \in D. \tag{2.28}$$

The next theorem, providing the main result for the Poisson equation on bounded domains, makes this statement more rigorous.

Theorem 2.29. *Let D be bounded and all points on ∂D be regular. Furthermore, let h be a C^2-function on $\mathbb{R}^n$ and f be a continuous function on $\mathbb{R}^n$. Then* (2.27) *holds if and only if* (2.28) *does.*

Proof. First, suppose that (2.27) holds. Since h does not need to have compact support, we take another C^2-function $\tilde{h}$ with compact support that agrees with h on D. Let $x \in D$. We can apply Proposition 1.14 to see that

$$\tilde{h}(B_t) - \int_0^t \frac{1}{2}\Delta\tilde{h}(B_s)\, ds, \qquad t \geq 0$$

is a martingale with respect to $\mathbb{P}^x$. By the stopping time theorem,

$$\tilde{h}(B_{\tau_D \wedge t}) - \int_0^{\tau_D \wedge t} \frac{1}{2} \Delta \tilde{h}(B_s)\, ds, \qquad t \geq 0$$

is also a martingale with respect to $\mathbb{P}^x$. Since $\tilde{h} = h$ on D, we can replace $\tilde{h}$ by h to get that

$$h(B_{\tau_D \wedge t}) - \int_0^{\tau_D \wedge t} \frac{1}{2} \Delta h(B_s)\, ds, \qquad t \geq 0$$

is also a martingale with respect to $\mathbb{P}^x$. Here, we had to pay attention to the case $\tau_D = 0$. We only could replace $\tilde{h}$ by h, if $B_0 = D$, i.e. $x \in D$. Taking this into account, let $t \geq 0$ and use (2.27) to get

$$\mathbb{E}^x \left[h(B_{\tau_D \wedge t}) - \int_0^{\tau_D \wedge t} \frac{1}{2} \Delta h(B_s)\, ds \right] = \mathbb{E}^x h(B_{\tau_D \wedge t}) + \mathbb{E}^x \int_0^{\tau_D \wedge t} f(B_s)\, ds.$$

On the other hand, the martingale property yields

$$\mathbb{E}^x \left[h(B_{\tau_D \wedge t}) - \int_0^{\tau_D \wedge t} \frac{1}{2} \Delta h(B_s)\, ds \right] = \mathbb{E}^x h(B_0) = h(x).$$

It follows

$$h(x) = \mathbb{E}^x h(B_{\tau_D \wedge t}) + \mathbb{E}^x \int_0^{\tau_D \wedge t} f(B_s)\, ds.$$

Note that τ_D is finite $\mathbb{P}^x$-a.s., since D is bounded. Furthermore, note that $h|_{\overline{D}}$ and $f|_{\overline{D}}$ are continuous and bounded, and $h \equiv 0$ on ∂D by assumption. Thus, we can apply the bounded convergence theorem for letting $t \to \infty$, which gives (2.28).

Now, assume (2.28). The plan is to split the right side of (2.28) into two summands using an exit time from a ball, then to rewrite the one summand introducing $f(x)$ and to expand the other summand into a Taylor series on the ball to bring the Laplace operator into play. So, take $x \in D$ and $r > 0$ such that $B(x,r) \subset D$. Split (2.28) into two summands and apply the strong Markov property to get

$$\begin{aligned} h(x) &= \mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(B_s)\, ds + \mathbb{E}^x \int_{\tau_{B(x,r)}}^{\tau_D} f(B_s)\, ds \\ &= \mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(B_s)\, ds + \mathbb{E}^x \left[\mathbb{E}^x \left[\int_{\tau_{B(x,r)}}^{\tau_D} f(B_s)\, ds \,\middle|\, \mathcal{F}_{\tau_{B(x,r)}} \right] \right] \\ &= \mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(B_s)\, ds + \mathbb{E}^x \left[\mathbb{E}^{B_{\tau_{B(x,r)}}} \int_0^{\tau_D} f(B_s)\, ds \right] \\ &= \underbrace{\mathbb{E}^x \int_0^{\tau_{B(x,r)}} f(B_s)\, ds}_{:=P_1} + \underbrace{\mathbb{E}^x h(B_{\tau_{B(x,r)}})}_{:=P_2}. \end{aligned} \tag{2.29}$$

We first care about P_1, which can, in a first step, be rewritten as

$$P_1 = \mathbb{E}^x \left[\int_0^{\tau_{B(x,r)}} f(B_s) - f(x)\, ds \right] + f(x)\mathbb{E}^x \tau_{B(x,r)}.$$

We want to prove that $f(B_s) - f(x) \in o(1)$ for $s \in \left[0, \tau_{B(x,r)}\right]$ letting $r \downarrow 0$, i.e. $f(B_s) - f(x) \xrightarrow{r \to 0} 0$ for $s \in \left[0, \tau_{B(x,r)}\right]$. For this, take a decreasing sequence $(r_n)_{n \in \mathbb{N}}$ with $r_n \downarrow 0$ as $n \to \infty$ and let $\epsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|f(y) - f(x)| \leq \epsilon$ for each $y \in \mathbb{R}^n$ with $|y - x| \leq \delta$. Choose $N \in \mathbb{N}$, such that $r_N \leq \delta$ implying $B_s \in B(x, r_N) \subseteq B(x, \delta)$ (i.e. $|B_s - x| \leq \delta$) for every $s \in \left[0, \tau_{B(x,r_N)}\right]$. Since r_n is decreasing, it follows

$$|f(B_s) - f(x)| \leq \epsilon \quad \text{for all } s \in \left[0, \tau_{B(x,r_n)}\right] \text{ and all } n \geq N.$$

This gives

$$P_1 = \left[f(x) + o(1)\right]\mathbb{E}^x \tau_{B(x,r)}.$$

Now, P_2. Expanding P_2 into a Taylor series on $B(x,r)$ at x up to second order gives

$$\begin{aligned} P_2 = h(x) &+ \underbrace{\mathbb{E}^x \left[\sum_{i=1}^n \frac{\partial h}{\partial x_i}(x) \left(B^i_{\tau_{B(x,r)}} - x_i\right) \right]}_{:=E} \\ &+ \underbrace{\mathbb{E}^x \left[\frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \left(B^i_{\tau_{B(x,r)}} - x_i\right) \left(B^j_{\tau_{B(x,r)}} - x_j\right) \right]}_{:=F} + o(r^2). \end{aligned}$$

The term E is zero, since $(B^i_t)_{t \geq 0}$ is a martingale (i.e. $\mathbb{E}^x B^i_{\tau_{B(x,r)}} = \mathbb{E}^x B^i_0 = x_i$). For F, it holds

$$F = \frac{1}{2}\left(\underbrace{\mathbb{E}^x \left[\sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(x) \left(B^i_{\tau_{B(x,r)}} - x_i\right)^2 \right]}_{:=F_1} + \underbrace{\mathbb{E}^x \left[\sum_{\substack{i,j=1 \\ i \neq j}}^n \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \left(B^i_{\tau_{B(x,r)}} - x_i\right) \left(B^j_{\tau_{B(x,r)}} - x_j\right) \right]}_{:=F_2} \right).$$

Since $((B^i_t)^2 - t)_{t \geq 0}$ is a martingale by Proposition 1.3 (a), we can rewrite F_1 as

$$\begin{aligned} F_1 &= \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(x) \mathbb{E}^x \left[\left(B^i_{\tau_{B(x,r)}}\right)^2 - \tau_{B(x,r)} + \tau_{B(x,r)} - 2B^i_{\tau_{B(x,r)}} x_i + x_i^2 \right] \\ &= \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(x) \left(x_i^2 + \mathbb{E}^x \tau_{B(x,r)} - 2x_i^2 + x_i^2 \right) \\ &= \Delta h(x) \mathbb{E}^x \tau_{B(x,r)}. \end{aligned}$$

Also recall that $(B^i_t B^j_t)_{t\geq 0}$ is a martingale for $i \neq j$ by Proposition 1.3 (b). Thus, for F_2 it follows

$$F_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j}(x)\mathbb{E}^x\left[B^i_{\tau_{B(x,r)}} B^j_{\tau_{B(x,r)}} - B^i_{\tau_{B(x,r)}} x_j - x_i B^j_{\tau_{B(x,r)}} + x_i x_j\right]$$
$$= \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j}(x)\left(x_i x_j - x_i x_j - x_i x_j + x_i x_j\right) = 0.$$

Summarizing the previous steps gives

$$P_2 = h(x) + \frac{1}{2}F_1 = h(x) + \frac{1}{2}\Delta h(x)\mathbb{E}^x \tau_{B(x,r)},$$

which is, by regarding (2.29), implying the following equalities:

$$\frac{1}{2}\Delta h(x)\mathbb{E}^x\tau_{B(x,r)} = -\Big[f(x) + o(1)\Big]\mathbb{E}^x\tau_{B(x,r)}$$
$$\iff \quad \frac{1}{2}\Delta h(x) = -\Big[f(x) + o(1)\Big]$$

Since $\mathbb{E}^x\tau_{B(x,r)} = \frac{r^2}{n} > 0$ by the remark after Example 2.12, $\frac{1}{2}\Delta h(x) = -f(x)$ follows by letting $r \downarrow 0$.

It remains to show that $\lim_{\substack{x\to z\\ x\in D}} h(x) = 0$ for each $z \in \partial D$. So, take $z \in \partial D$. Since f is continuous on $\mathbb{R}^n$, it is bounded on $\overline{D}$. This means, it suffices to show that

$$\lim_{\substack{x\to z\\ x\in D}} \mathbb{E}^x\tau_D = 0. \tag{2.30}$$

In a first step, we compute for $t > 0$ by using the Cauchy-Schwarz inequality

$$\begin{aligned} \mathbb{E}^x\tau_D &= \mathbb{E}^x\left[\tau_D, \tau_D \leq t\right] + \mathbb{E}^x\left[\tau_D, \tau_D > t\right] \\ &\leq t\,\mathbb{P}^x\left(\tau_D \leq t\right) + \mathbb{E}^x\left[\tau_D^2\right]^{1/2}\mathbb{P}^x\left(\tau_D > t\right)^{1/2}. \end{aligned} \tag{2.31}$$

Note that, since D is bounded, $\mathbb{E}^x\tau_D < \infty$, and hence $\mathbb{E}^x\tau_D^2$ is uniformly bounded for $x \to z$, $x \in D$ by Proposition 2.30, which is proved afterwards. Indeed, find $r > 0$ with $D \subset B(0,r)$ and compute

$$\mathbb{E}^x\tau_D^2 \leq \left(\frac{\sqrt{\mathbb{E}^x B^4_{\tau_D}} + x^2}{3 - \sqrt{6}}\right)^2 \leq \left(\frac{r^2 + x^2}{3 - \sqrt{6}}\right)^2.$$

By Lemma 2.20, we see that

$$\liminf_{\substack{x\to z\\ x\in D}} \mathbb{P}^x\left(\tau_D \le t\right) \ge \mathbb{P}^z\left(\tau_D \le t\right) = 1,$$

where we used the regularity of z in the last step. It follows that

$$\lim_{\substack{x\to z\\ x\in D}} \mathbb{P}^x\left(\tau_D \le t\right) = 1 \text{ and } \lim_{\substack{x\to z\\ x\in D}} \mathbb{P}^x\left(\tau_D > t\right) = 0.$$

Thus, taking the limit $x \to z$, $x \in D$ in (2.31) yields to

$$\lim_{\substack{x\to z\\ x\in D}} \mathbb{E}^x \tau_D \le t.$$

Since $t > 0$ was arbitrary, we get (2.30), so the proof is complete. ■

Proposition 2.30. *Let $x \in D$ and τ be a stopping time with $\mathbb{E}^x\tau < \infty$. Then*

$$\mathbb{E}^x\tau^2 \le \left(\frac{\sqrt{\mathbb{E}^x B_\tau^4} + x^2}{3-\sqrt{6}}\right)^2.$$

Proof. Recall from Proposition 1.3 (c) that $(B_t^4 - 6tB_t^2 + 3t^2)_{t\ge 0}$ is a martingale under $\mathbb{P}^x$. Let $t \ge 0$ and see that by the stopping time theorem

$$\mathbb{E}^x B_{\tau\wedge t}^4 - 6\mathbb{E}^x\left[(\tau\wedge t)B_{\tau\wedge t}^2\right] + 3\mathbb{E}^x\left[(\tau\wedge t)^2\right] = x^4.$$

Rearranging and applying the Cauchy-Schwarz inequality gives

$$\mathbb{E}^x B_{\tau\wedge t}^4 + 3\mathbb{E}^x\left[(\tau\wedge t)^2\right] \le 6\left(\mathbb{E}^x\left[(\tau\wedge t)^2\right]\mathbb{E}^x B_{\tau\wedge t}^4\right)^{1/2} + x^4.$$

For $u := \sqrt{\mathbb{E}^x B_{\tau\wedge t}^4}$ and $v := \sqrt{\mathbb{E}^x\left[(\tau\wedge t)^2\right]}$, this is

$$u^2 - 6uv + 3v^2 \le x^4$$

implying

$$\begin{aligned}
& & (3v-u)^2 &\le x^4 + 6v^2\\
&\Longrightarrow & 3v - u &\le \sqrt{x^4+6v^2} \le x^2 + \sqrt{6}v\\
&\Longleftrightarrow & v &\le \frac{u+x^2}{3-\sqrt{6}}\\
&\Longleftrightarrow & v^2 &\le \left(\frac{u+x^2}{3-\sqrt{6}}\right)^2. \qquad (2.32)
\end{aligned}$$

Since $(B_{\tau\wedge t})_{t\geq 0}$ is a martingale, we use the Jensen inequality to get

$$\mathbb{E}^x B_{\tau\wedge t}^4 = \mathbb{E}^x \left[\mathbb{E}^x \left[B_\tau \mid \mathcal{F}_{\tau\wedge t}\right]^4\right] \leq \mathbb{E}^x B_\tau^4.$$

We conclude from this and (2.32) that

$$\mathbb{E}^x \left[(\tau \wedge t)^2\right] \leq \left(\frac{\sqrt{\mathbb{E}^x B_\tau^4} + x^2}{3 - \sqrt{6}}\right)^2 .$$

Letting $t \uparrow \infty$ completes the proof, since $\mathbb{E}^x \tau < \infty$. ■

Appendix

A.1 Notation

Here are some helpful notations being used throughout the text:

(a) Sets of numbers:

(i) $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$

(ii) $\mathbb{Q} \mathrel{\hat{=}}$ rational numbers

(iii) $\mathbb{R} \mathrel{\hat{=}}$ real numbers, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$

(iv) $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$

(b) $a \wedge b := \min\{a, b\}$

(c) For an event $A \in \mathcal{F}$, the indicator function of A is denoted by $\mathbb{1}_A$.

(d) $\mathbb{E}X$ denotes the expectation value of a random variable X.

(e) $a_n \uparrow\downarrow a$ means that the sequence $(a_n)_{n\in\mathbb{N}}$ is increasingly, respectively decreasingly, converging to a as $n \to \infty$.

(f) A function f is said to be C^k if f is k times continuously differentiable on its domain.

A.2 Analysis

In the following, let $D \subset \mathbb{R}^n$ be bounded and open. ∂D is assumed to be sufficiently regular with surface measure dS and outward normal vector ν.

Notation. For a (twice) differentiable function u on D and $i, j \in [n]$, define

$$u_{x_i} := \frac{\partial u}{\partial x_i}, \qquad u_{x_i x_j} := \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Theorem A.1 (Gauss-Green Theorem). *Let $u \in C^1(\overline{D})$. Then*

$$\int_D u_{x_i}\,dx = \int_{\partial D} u\nu^i\,dS, \qquad i \in [n].$$

Theorem A.2 (Green's Formula). *Let $u \in C^1(\overline{D})$. Then*

$$\int_D \Delta u\,dx = \int_{\partial D} \nabla u \cdot \nu\,dS.$$

Proof. For $i \in [n]$ apply Theorem A.1 to u_{x_i} to get

$$\int_D u_{x_i x_i}\,dx = \int_{\partial D} u_{x_i}\nu^i\,dS.$$

Now sum for $i \in [n]$ to get the result. ∎

Theorem A.3 (Spherical coordinates). *Let $u \in C(D)$ be an integrable function. For all $x \in D$ and $r > 0$ such that $B(x,r) \subset D$, it holds*

$$\int_{B(x,r)} u(y)\,dy = \int_0^r \left(\int_{\partial B(x,s)} u(y)\,dS \right) ds.$$

Lemma A.4. *Let $u \in C(D)$ and $x \in D$. Then it holds*

$$\int_{\partial B(x,r)} u(y)\,\sigma_{x,r}(dy) \xrightarrow{r \to 0} u(x).$$

Proof. Let $\epsilon > 0$. Since u is continuous, there exists an $r > 0$ such that

$$|y - x| \le r \Rightarrow |u(y) - u(x)| \le \epsilon.$$

Thus the following calculation shows the result:

$$\begin{aligned}
\left| \int_{\partial B(x,r)} u(y)\,\sigma_{x,r}(dy) - u(x) \right| &= \left| \int_{\partial B(x,r)} u(y)\,\sigma_{x,r}(dy) - \int_{\partial B(x,r)} u(x)\,\sigma_{x,r}(dy) \right| \\
&\le \int_{\partial B(x,r)} |u(y) - u(x)|\,\sigma_{x,r}(dy) \\
&\le \epsilon
\end{aligned}$$

∎

Lemma A.5. *Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of continuous functions on $\mathbb{R}^n$ such that $f_n \xrightarrow{n\to\infty} f$ pointwise. Then f is lower semicontinuous on $\mathbb{R}^n$, i.e.*

$$\liminf_{x \to z} f(x) \ge f(z)$$

for all $z \in \mathbb{R}^n$.

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is increasing, it holds that

$$f(x) \geq f_m(x)$$

for every $m \in \mathbb{N}$ and every $x \in \mathbb{R}^n$. Fix $z \in \mathbb{R}^n$ and $m \in \mathbb{N}$. It follows

$$\liminf_{x\to z} f(x) \geq \liminf_{x\to z} f_m(x) = f_m(z).$$

Letting $m \to \infty$ gives

$$\liminf_{x\to z} f(x) \geq f(z).$$

■

A.3 Measure theory

Definition A.6. Let $(\Omega, \mathcal{F})$ be a measurable space.

(a) A set of events $\mathcal{P} \subset \mathcal{F}$ is called a *π-system*, if

$$A, B \in \mathcal{P} \quad \text{implies} \quad A \cap B \in \mathcal{P}.$$

(b) A set of events $\mathcal{L} \subset \mathcal{F}$ is said to be a λ-system, if

(i) $\Omega \in \mathcal{L}$,

(ii) $A, B \in \mathcal{L}$ and $A \subset B$ implies $B \backslash A \in \mathcal{L}$,

(iii) $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ and $A_n \uparrow A$ implies $A \in \mathcal{L}$.

Theorem A.7 (π-λ-theorem)**.** *Let $\mathcal{P}$ be a π-system and $\mathcal{L}$ be a λ-system with $\mathcal{P} \subset \mathcal{L}$. Then $\sigma(\mathcal{P}) \subset \mathcal{L}$.* [4, **Theorem A.1.4 (p.402)**]

Theorem A.8 (Monotone class theorem)**.** *Let $\mathcal{P}$ be a π-system with $\Omega \in \mathcal{P}$ and let $\mathcal{H}$ be a vector space of random variables satisfying the following properties:*

(i) $A \in \mathcal{P}$ implies $\mathbb{1}_A \in \mathcal{H}$.

(ii) $(X_n)_{n\in\mathbb{N}} \subset \mathcal{H}$, X bounded and $X_n \uparrow X$ implies $X \in \mathcal{H}$.

Then $\mathcal{H}$ contains all bounded $\sigma(\mathcal{P})$-measurable random variables.

Proof. Define

$$\mathcal{L} := \{A \in \mathcal{F} : \mathbb{1}_A \in \mathcal{H}\}.$$

We show that $\mathcal{L}$ is a λ-system by proving the three defining properties from above:

1. $\Omega \in \mathcal{L}$, since $\Omega \in \mathcal{P}$ by assumption and (i).
2. Take $A, B \in \mathcal{L}$ with $B \subset A$. It holds

   $$\mathbb{1}_{B\backslash A} = \mathbb{1}_B - \mathbb{1}_A \in \mathcal{H},$$

   since $\mathcal{H}$ is a vector space. So, $B\backslash A \in \mathcal{L}$.
3. Take $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ with $A_n \uparrow A$. It holds

   $$\mathbb{1}_A = \lim_{n\to\infty} \mathbb{1}_{A_n} \in \mathcal{H}$$

   by (ii). So, $A \in \mathcal{L}$.

We apply the π-λ-theorem, Theorem A.7, to $\mathcal{P}$ and $\mathcal{L}$ to get

$$\sigma(\mathcal{P}) \subset \mathcal{L}$$

expressing that $\mathbb{1}_A \in \mathcal{H}$ for all $A \in \sigma(\mathcal{P})$. Since $\mathcal{H}$ is a vector space of random variables, $\mathcal{H}$ also contains all simple $\sigma(\mathcal{P})$-measurable random variables. Finally, by (ii) we get that $\mathcal{H}$ contains all bounded $\sigma(\mathcal{P})$-measurable random variables. ■

A.4 Probability theory

Lemma A.9. *Let A and B be two events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(B) = 1$. Then it holds*

$$\mathbb{P}(A) = \mathbb{P}(A \cap B).$$

Proof. Since

$$\mathbb{P}(A\backslash B) = \underbrace{\mathbb{P}(A \cup B)}_{\geq \mathbb{P}(B)=1} - \mathbb{P}(B) = 1 - 1 = 0,$$

we compute

$$\mathbb{P}(A) = \mathbb{P}(A\backslash B) + \mathbb{P}(A \cap B) = \mathbb{P}(A \cap B).$$

■

Theorem A.10 (Uniqueness Theorem for Characteristic Functions)**.** *If two random vectors have the same characteristic function, then they have the same distribution.* [4, **Theorem 3.9.4 (p.176)**]

Proposition A.11. *Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that f is a bounded measurable function on $\mathbb{R}^2$ and $\mathcal{G}$ is a σ-algebra. If X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$, then*

$$\mathbb{E}\left[f(X,Y)\,|\,\mathcal{G}\right] = g(X) \quad \mathbb{P}\text{-a.s.},$$

where $g(x) := \mathbb{E}f(x,Y)$.

Proof. First, since X is $\mathcal{G}$-measurable and g is measurable, $g(X)$ is $\mathcal{G}$-measurable. It remains to show that

$$\mathbb{E}\left[f(X,Y), A\right] = \mathbb{E}\left[g(X), A\right]$$

for each $A \in \mathcal{G}$. So, take $A \in \mathcal{G}$ and note that $(X, \mathbb{1}_A)$ is independent of Y. Let μ be the distribution of $(X, \mathbb{1}_A)$ and ν be the distribution of Y. We get

$$\begin{aligned}\mathbb{E}\left[f(X,Y), A\right] &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(x,y) \cdot a \, d\mu(x,a) \, d\nu(y) \\ &\overset{(*)}{=} \int_{\mathbb{R}^2} \underbrace{\int_{\mathbb{R}} f(x,y) \, d\nu(y)}_{=g(x)} \, d\mu(x,a) \\ &= \mathbb{E}\left[g(X), A\right],\end{aligned}$$

where we used Fubini's theorem at $(*)$. ■

Theorem A.12 (Convergence theorem for discrete time martingales)**.** *If $(M_n)_{n\in\mathbb{N}}$ is a martingale that satisfies $\sup_{n\in\mathbb{N}} \mathbb{E}|M_n| < \infty$, then $\lim_{n\to\infty} M_n$ exists and is $\mathbb{P}$-a.s. finite. Furthermore, if the martingale is uniformly integrable, then the convergence also occurs in L^1.* [4, **Theorem 5.2.8 (p.236)**]

Theorem A.13 (Convergence theorem for continuous time martingales)**.** *If $(M_t)_{t\geq 0}$ is a right-continuous martingale that satisfies $\sup_{t\geq 0} \mathbb{E}|M_t| < \infty$, then $\lim_{t\to\infty} M_t$ exists and is $\mathbb{P}$-a.s. finite. Furthermore, if the martingale is uniformly integrable, then the convergence also occurs in L^1.* [1, **Theorem 1.92 (p.39)**]

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